

# LECTURES ON FACTORIZATION OF BIRATIONAL MAPS

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*in memory of Prof. Kunihiko Kodaira*

This is an expanded version of the notes for the lectures given by the author at RIMS in the summer of 1999 to provide a detailed account of the proof for the (weak) factorization theorem of birational maps by Abramovich-Karu-Matsuki-Włodarczyk [1]. All the main ideas of these lecture notes, therefore, are derived from the collaboration and joint work with D. Abramovich, K. Karu and J. Włodarczyk. The author would like to emphasize that the theory of “Birational Cobordism”, which plays the central role in our proof, was developed in Włodarczyk [2] and that its origin may be traced back to the earlier work of Morelli [1][2] in his brilliant solution to the (strong) factorization conjecture for toric birational maps. The author would also like to emphasize that there is another independent proof for the (weak) factorization theorem by Włodarczyk [3], which is NOT discussed in these notes. It is noteworthy that not only the ideas of but also some essential steps of the two proofs rely on the work of Morelli: our proof in Abramovich-Karu-Matsuki-Włodarczyk [1] reduces the factorization of general birational maps to that of toroidal birational maps and then uses the combinatorial algorithm by Morelli to factor toroidal ones. The proof in Włodarczyk [2] uses the process of “ $\pi$ -desingularization”, which is the most difficult and subtle part of the algorithm by Morelli. In fact, all that we cover in these notes is the above mentioned reduction step “from general to toroidal”, taking the combinatorial algorithm for the factorization of toroidal birational maps as a black box. We refer the reader to Abramovich-Matsuki-Rashid [1] for what is inside of the black box, including the details of the process of  $\pi$ -desingularization and clarifications of some discrepancies contained in the original arguments of Morelli [1][2].

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## CHAPTER 0. INTRODUCTION

The purpose of these lecture notes is to provide a detailed account of the proof given by Abramovich-Karu-Matsuki-Włodarczyk [1] for the following (weak) factorization conjecture for birational maps. (We note that there is another proof independently given by Włodarczyk [1], which we will NOT discuss in these notes.)

**Weak Factorization Theorem of Birational Maps.** Let  $\phi : X_1 \dashrightarrow X_2$  be a birational map between complete nonsingular varieties over an algebraically closed field  $K$  of characteristic zero. Let  $X_1 \supset U \subset X_2$  be a common open subset over which  $\phi$  is an isomorphism. Then  $\phi$  can be factored into a sequence of blowups and blowdowns with smooth irreducible centers disjoint from  $U$ . That is to say, there exists a sequence of birational maps between complete nonsingular varieties

$$X_1 = V_1 \xrightarrow{\psi_1} V_2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{i-1}} V_i \xrightarrow{\psi_i} V_{i+1} \xrightarrow{\psi_{i+1}} \cdots \xrightarrow{\psi_{l-2}} V_{l-1} \xrightarrow{\psi_{l-1}} V_l = X_2$$

where

- (i)  $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$ ,
- (ii)  $\psi_i$  are isomorphisms over  $U$ , and
- (iii) either  $\psi_i : V_i \dashrightarrow V_{i+1}$  or  $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$  is a morphism obtained by blowing up a smooth irreducible center disjoint from  $U$ .

Furthermore, if both  $X_1$  and  $X_2$  are projective, then we can choose a factorization so that all the intermediate varieties  $V_i$  are projective. More precisely, in the factorization we construct, there exists an index  $i_o$  such that the birational map  $V_i \dashrightarrow X_1$  is a projective birational morphism for  $i \leq i_o$  and that the birational map  $V_i \dashrightarrow X_2$  is a projective birational morphism for  $i_o \leq i \leq l$ .

It is called the WEAK factorization theorem as we allow the sequence to consist of blowups and blowdowns in any order. If we insist on the sequence in the factorization to consist only of blowups immediately followed by blowdowns, then we obtain the following STRONG factorization conjecture, which remains open.

**Strong Factorization Conjecture.** Let  $\phi : X_1 \dashrightarrow X_2$  be as above. Then a sequence as above can be chosen with the extra condition that there exists an index  $i_o$  so that  $\psi_i^{-1}$  for  $1 \leq i \leq i_o - 1$  are blowups of  $V_i$  while  $\psi_i$  for  $i_o \leq i \leq l - 1$  are blowdowns of  $V_i$ .

We note that, if both  $X_1$  and  $X_2$  are projective, in a sequence asserted in the strong factorization conjecture all the intermediate varieties are automatically projective, while in an arbitrary sequence in the weak factorization it may have some complete but nonprojective intermediate varieties and that the “Moreover” part of the weak factorization theorem asserts the existence of a sequence preserving the projectivity.

Several generalizations of the theorem will be discussed in Chapter 5.

### §0-1. Outline of the strategy

First, we simply state the strategy of our proof in a symbolic way with a brief description of the following five main steps:

Strategy of the Proof

Step 1: Reduction to the case where  $\phi$  is a projective birational morphism

#### Elimination of points of indeterminacy

Step 2: Factorization into locally toric birational maps

### **Construction of a birational cobordism**

Step 3: Factorization into toroidal birational maps

### **Torification : Blowing up the torific ideal**

Step 4: Recovery of Nonsingularity

### **Canonical resolution of singularities**

Step 5. Factorization of toroidal birational maps among nonsingular toroidal embeddings

### **Morelli's combinatorial algorithm.**

Now we will explain each step more in detail and describe the contents of the chapters of these notes accordingly.

Let  $\phi : X_1 \dashrightarrow X_2$  be a birational map between complete nonsingular varieties over an algebraically closed field  $K$  of characteristic zero.

Step 1: Reduction to the case where  $\phi$  is a projective birational morphism

Hironaka's elimination of points of indeterminacy asserts that there exist sequences of blowups with smooth centers  $X'_1 \rightarrow X_1$  and  $X'_2 \rightarrow X_2$  so that the induced map  $\phi' : X'_1 \rightarrow X'_2$  is a projective birational morphism. By replacing the original  $\phi$  with  $\phi'$  we may assume in the factorization problem that  $\phi$  is a projective birational morphism. This process is discussed in §1-4 of Chapter 1.

Step 2: Factorization into locally toric birational maps

This is the key step of our proof, discussed in Chapter 2, which allows us to bring in a "Morse Theoretic" view point to approach the factorization problem via the theory of "Birational Cobordism".

Let us briefly recall what the Morse theory tells us: We have a usual cobordism  $C = C(M_1, M_2)$  between differentiable manifolds  $M_1$  and  $M_2$  equipped with a Morse function  $f : C(M_1, M_2) \rightarrow \mathbb{R}$ . (By extending the cobordism appropriately, we may assume that  $f$  is surjective.)

**Picture 0-1-1.**

By looking at the vector field  $\text{grad}(f)$  (with respect to a Riemannian metric on  $C(X_1, X_2)$ ), we can introduce the action of the “time”  $t \in \mathbb{R}$  on the cobordism, namely, we have a map from the time  $\mathbb{R}$  to the group  $\text{Diff}(C)$  of diffeomorphisms of  $C$  denoted by

$$t \in \mathbb{R} \mapsto \varphi_t : C \rightarrow C \in \text{Diff}(C)$$

where  $\varphi_t(p)$  is the position of a point  $p \in C$  after time  $t$  along the flow given by integrating the vector field  $\text{grad}(f)$ . Taking the exponential  $\exp(t) \in \mathbb{R}_{>0}$  of the time  $t \in \mathbb{R}$ , we see that the multiplicative group  $(\mathbb{R}_{>0}, \times) = \exp(\mathbb{R}, +)$  acts as well. The crucial information in the Morse Theory can be read off in terms of the action of the multiplicative group:

(C-o) the multiplicative group  $\mathbb{R}_{>0}$  acts on a cobordism  $C$ ,

(C-i)  $M_2$  and  $M_1$  being on the top and at the bottom of the cobordism  $C$ , respectively, we can describe them as the quotients

$$\begin{aligned} M_2 &\cong C_+ / \mathbb{R}_{>0} \\ M_1 &\cong C_- / \mathbb{R}_{>0} \end{aligned}$$

where the sets  $C_+$  and  $C_-$  are defined to be

$$C_+ := \{x \in C; \lim_{t \rightarrow \infty} t(x) = \lim_{\exp(t) \rightarrow \infty} \exp(t)(x) \text{ does NOT exist in } C\}$$

$$C_- := \{x \in C; \lim_{t \rightarrow -\infty} t(x) = \lim_{\exp(t) \rightarrow 0} \exp(t)(x) \text{ does NOT exist in } C\},$$

(C-ii) the critical points of  $f$  are precisely the fixed points of the action and the homotopy types of fibers of  $f$  change as we pass through these fixed points.

This interpretation of the Morse Theory in terms of the action of the multiplicative group led Włodarczyk [2] naturally to the notion of a birational cobordism  $B = B_\phi(X_1, X_2)$  for a birational map  $\phi$ :

(B-o) the multiplicative group  $t \in K^*$  acts on a (nonsingular) variety  $B$  of  $\dim B = \dim X_1 + 1 = \dim X_2 + 1$ ,

(B-i) we can describe  $X_2$  and  $X_1$  as the quotients

$$\begin{aligned} X_2 &\cong B_+/K^* \\ X_1 &\cong B_-/K^* \end{aligned}$$

where the sets  $B_+$  and  $B_-$  are defined to be

$$\begin{aligned} B_+ &:= \{x \in B; \lim_{t \rightarrow \infty} t(x) \text{ does NOT exist in } B\} \\ B_- &:= \{x \in B; \lim_{t \rightarrow 0} t(x) \text{ does NOT exist in } B\} \end{aligned}$$

and the birational map induced by the natural inclusions

$$X_1 = B_+/K^* \supset (B_+ \cap B_-)/K^* \subset B_-/K^* = X_2$$

coincides with  $\phi$ ,

(B-ii) “passing through” the fixed points of the action induces the birational transformations.

The precise definition of the birational cobordism is found in §2-1. We give the construction of a birational cobordism (only for a projective birational morphism, after the reduction Step 1, and leaving the construction for a general birational map to the original Włodarczyk [2]) in §2-2, together with the discussion of the “collapsibility” in order to line up the fixed points nicely in the above algebraic setting where we lack a Morse function  $f$ , which, in the usual differentiable setting, would assign the levels to the critical points and hence line them up nicely. We also provide an interpretation of the notion of the birational cobordism in §2-3 from a view point of Geometric Invariant Theory (reviewed briefly in §1-1 together with Toric Geometry), in connection with the work of Thaddeus and others. This view point allows us to preserve the projectivity of the intermediate varieties in our factorization if both  $X_1$  and  $X_2$  are projective.

We would like to take a closer look at the birational transformations as we pass through the fixed points.

Let  $p \in B^{K^*}$  be a fixed point of the birational cobordism  $B$ . Analytically locally, the action of  $K^*$  on  $B$  at  $p$  is equivalent to the action of  $K^*$  on the tangent space  $T_{B,p}$ , which is a vector space  $V_p$  of  $\dim V_p = n$ . As the multiplicative group  $K^*$  is reductive, the vector space  $V_p$  splits into the eigenspaces according to the characters of the action. By choosing a basis  $\{z_1, \dots, z_n\}$  from the eigenspaces, we can describe the action of  $t \in K^*$  on  $T_{B,p} = V_p$  as

$$t(z_1, \dots, z_n) = (t^{\alpha_1} \cdot z_1, \dots, t^{\alpha_n} \cdot z_n)$$

where the  $\alpha_i \in \mathbb{Z}$  are the characters for the  $z_i$ . The crucial sets  $(V_p)_+$  and  $(V_p)_-$  for the local behavior of the cobordism, defined similarly as  $B_+$  and  $B_-$ , have simple descriptions:

$$\begin{aligned} (V_p)_+ &= \{z = (z_1, \dots, z_n) \in V_p; z_i \neq 0 \text{ for some } i \text{ with } \alpha_i > 0\} \\ (V_p)_- &= \{z = (z_1, \dots, z_n) \in V_p; z_i \neq 0 \text{ for some } i \text{ with } \alpha_i < 0\}. \end{aligned}$$

If we identify the vector space  $V_p$  with the affine  $n$ -space  $\mathbb{A}^n = X(N, \sigma)$ , regarded as the toric variety associated to the standard cone  $\sigma = \langle v_1, \dots, v_n \rangle \subset N_{\mathbb{R}} = N \otimes \mathbb{R}$  generated by the standard  $\mathbb{Z}$ -basis of the lattice  $N \cong \mathbb{Z}^n$ , and identify the action of  $K^*$  with the one-parameter subgroup corresponding to the point  $a = (\alpha_1, \dots, \alpha_n) \in N$ , we have a simple local description of the birational transformation, namely the birational transformation between  $(V_p)_+/K^*$  and  $(V_p)_-/K^*$  in terms of toric geometry associate to the projection  $\pi : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R} \cdot a$ ;

$$\begin{aligned}(V_p)_+/K^* &= X(\pi(N), \pi(\partial_+\sigma)) \\ (V_p)_-/K^* &= X(\pi(N), \pi(\partial_-\sigma))\end{aligned}$$

where the upper boundary face and lower boundary face of  $\sigma$  are defined to be

$$\begin{aligned}\partial_+\sigma &:= \{v \in \sigma; v + \epsilon \cdot (-a) \notin \sigma \text{ for } \epsilon > 0\} \\ \partial_-\sigma &:= \{v \in \sigma; v + \epsilon \cdot a \notin \sigma \text{ for } \epsilon > 0\}.\end{aligned}$$

Therefore, analytically locally, the birational transformation induced by passing through the fixed point  $p \in B$  is equivalent to a toric birational transformation between toric varieties corresponding to (the projections of) the upper boundary face and lower boundary face of the standard cone  $\sigma$  (with respect to the one-parameter subgroup  $a \in N$ ), which is a typical example of the “polyhedral cobordism” used by Morelli to solve the factorization problem for toric birational maps. Thus via the birational cobordism we succeed in factoring  $\phi$  into locally toric birational transformations as we pass through the fixed points (at the several levels ordered nicely by the collapsibility). This is the content of §2-4.

**Picture 0-1-2.**

One might naively expect at this point that since  $\phi$  is now factored into “locally toric” birational maps and since “toroidal” birational maps can be factored by Morelli’s combinatorial algorithm, we had already completed a proof for the weak factorization theorem, especially when the notion of “locally toric” and that of “toroidal” are mixed and sometimes confused in the existing literatures. But there is a huge GAP between locally toric birational maps and toroidal birational maps. Actually the distinction between the two notions is one of the key issues in our argument and the main bodies §1-2 and §1-3 of Chapter 1 are devoted to the detailed discussion and clarification of them.

### Step 3: Factorization into toroidal birational maps

The birational cobordism  $B$  has a locally toric structure, as we observed above, each point  $p \in B$  having an analytic neighborhood isomorphic to a toric variety  $V_p$ , and the action of  $K^*$  is translated into that of a one-parameter subgroup in this locally toric chart. But the choice of coordinates (or equivalently the coordinate divisors) is not canonical and hence the embedded tori vary from chart to chart as we move from point to point. Thus these locally toric coordinates do NOT patch together. On the other hand, we require a toroidal structure  $(U_X, X)$  to have a fixed open subset  $U_X \subset X$  so that  $X - U_X$  provides the global coordinate divisors. Thus in order to give  $B$  (or its modification) a toroidal structure we have to create such global divisors. “Torification” achieves exactly this goal by blowing up the “torific” ideal and hence creating these desired global coordinate divisors defined by the pull-back of the torific ideal. The torific ideal is canonically defined in terms of the action of  $K^*$  only.

Here arises one technical but essential difficulty. The torific ideal is only well-defined over smaller pieces of the birational cobordism, called the quasi-elementary pieces of  $B$ , but not on the entire birational cobordism  $B$ . (The quasi-elementary pieces correspond to the semi-stable loci of different linearizations of the action of  $K^*$ , when the birational cobordism is interpreted via Geometric Invariant Theory.) Thus we can only introduce toroidal structures, after blowing up the torific ideals, to the quasi-elementary pieces and these toroidal structures may not be compatible with each other. As a consequence, though we succeed in factoring  $\phi$  into toroidal birational maps by torification, we have to pay the price that these toroidal birational maps are only enough to provide weak factorization but not strong factorization, due to the incompatibility of the toroidal structures.

The precise definition of the torific ideal, together with the subtlety why it is only well-defined over the quasi-elementary pieces, is given in §3-1. The core of “torification”, i.e., the torifying property of the torific ideal asserting that blowing it up induces a toroidal structure so that the action of  $K^*$  is that of a one-parameter subgroup in a toroidal chart of each point, is discussed in §3-2.

### Step 4: Recovery from singular to nonsingular

Now that we succeeded in factoring  $\phi$  into toroidal birational maps after Step 3, are we done with the proof ? Not yet !!

Though the birational maps are toroidal, the toroidal embeddings we have constructed through the steps may be, and in most of the cases they actually are, SINGULAR. (In fact, even at the stage of factoring  $\phi$  into locally toric birational maps the intermediate varieties may be singular.) Since we seek to factor  $\phi$  into

blowups and blowdowns with smooth centers on nonsingular varieties, we have to bring back the whole situation from SINGULAR to NONSINGULAR. This is done via the application of the canonical resolution of singularities (and the canonical principalization of some ideals) in §4-1. The canonicity of the resolution allows us to preserve the toroidal structure through the process of desingularization.

**Step 5.** Factorization of toroidal birational maps among nonsingular toroidal embeddings

Finally, applying Morelli's combinatorial algorithm to toroidal birational maps between nonsingular toroidal embeddings, we complete the proof of the weak factorization in §4-2.

In Chapter 5, we discuss several generalizations of the weak factorization theorem, modifying the arguments discussed above according to the different purposes. §5-1 accomplishes the factorization of bimeromorphic maps in the analytic category. In §5-2, we consider the case where there is a group acting and establish the factorization equivariantly. (We remark that we may have to blow up several smooth centers simultaneously to maintain the equivariance.) As an application in §5-3, we show that we can remove the assumption of the base field  $K$  being algebraically closed. It is noteworthy that we can NOT remove the assumption of the characteristic of  $K$  being zero for the time being, as the canonical resolution of singularities sits in the center of our method, and that it is the only place where we use the characteristic assumption and hence any future development of the method of the canonical resolution of singularities in positive characteristics would allow us to remove the assumption. §5-4 discusses the factorization in the logarithmic category, which has an application to the study of the behavior of the Hodge structures under birational transformations.

In Chapter 6, we mention some problems related to our proof of the weak factorization conjecture. Though the proof, providing a specific procedure to factor a birational map, is constructive, it falls short of being effective at several places, of which §6-1 make a list. In §6-2, we discuss briefly the toroidalization conjecture, which lies at the heart of the whole circle of ideas, inspired by the work of De Jong [1] and others, involving the factorization problem, resolution of singularities, semi-stable reductions, the log categories of Kato [1] and beyond.

## §0-2. Some historical remarks

The following is the author's personal and prejudiced view of how the factorization problem evolved over years. It is by NO means meant to be a complete account of the subject and the author apologizes in advance for the omission of any references or topics which should be included.

### Early origins of the problem

A birational map  $\phi : X_1 \dashrightarrow X_2$  between algebraic varieties is by definition a rational map which induces an isomorphism of the function fields  $\phi^* : K(X_2) \xrightarrow{\sim} K(X_1)$  or equivalently a rational map which induces an isomorphism over a common dense open subset  $X_1 \supset U \subset X_2$ .

If we restrict our attention to complete nonsingular varieties, a birational map

is nothing but an isomorphism for curves, i.e., varieties of dimension 1. In dimension 2 or higher, however, the picture changes drastically. There are many examples of birational maps which are not isomorphisms, typical examples of which are BLOWUPS with smooth centers, their inverses BLOWDOWNS and their composites. A natural and fundamental question arises then if these exhaust all the possible birational maps, i.e., if a given birational map is a composite of blowups and blowdowns with smooth centers, which came to be known as the factorization problem of birational maps.

The history of the factorization problem of birational maps could be traced back to the Italian school of algebraic geometers, who already knew that the operation of blowing up points on surfaces is a fundamental source of richness for surface geometry. The importance of the strong factorization theorem in dimension 2 by Zariski cannot be overestimated in the analysis of the birational geometry of algebraic surfaces. The strong factorization problem was stated in the form of a question as “Question ( $F'$ )” in Hironaka [2], who recognized the connection with the problem of resolution of singularities and gave a partial answer in the form of Elimination of Points of Indeterminacy. Though the question of the weak factorization was also raised in Oda [1], the problem remained largely open in higher dimensions despite the efforts and interesting results of many, e.g., Crauder [1] Kulikov [1] Schaps [1] and others. These were summarized in Pinkham [1], where the weak factorization conjecture is explicitly stated in the form presented here.

### Toric Case

For toric birational maps, the equivariant version of the factorization problem under the torus action was posed in Oda [1] and came to be known as Oda’s weak and strong conjectures. Though the toric geometry can be interpreted in terms of the geometry of convex cones and hence sometimes is considered to be a “baby” or easier version of the real case, the factorization problem presented a substantial challenge and difficulty in combinatorics. (Actually the fact that all the known examples (cf. Hironaka [1] Sally [1] Shannon [1]) demonstrating the difficulties in higher dimensions were toric made some of us even suspect that it is not a coincidence but an indication that all the combinatorial complications in general are somehow concentrated in the toric case.) In dimension 3, Danilov [2]’s proof for the weak factorization was later supplemented by Ewald [1]. Oda’s weak conjecture was solved in arbitrary dimension by Włodarczyk [1]. A big breakthrough was brought by Morelli [1][2], who not only solved Oda’s strong conjecture in arbitrary dimension but also introduced the notion of a “cobordism” between fans into the analysis. This theory of the polyhedral cobordism made the combinatorial approach to the problem conceptually transparent, so much so that it made us eager to hope for the extension of the theory to be applied to general birational maps. But because of the apparent disguise as a purely combinatorial object with little hope of extension, we had to wait for Włodarczyk [2] to reveal the true meaning as the algebraic version of the Morse Theory.

### Minimal Model Program

It is worthwhile to note the relation of the factorization problem to the development of the Mori program. Hironaka [1] used the cone of effective curves to study the properties of birational morphisms. This direction was further developed and

given a decisive impact by Mori [2], who, motivated by Kleiman's criterion for ampleness and inspired by the solution to a conjecture of Hartshorne (cf. Mori [1]), introduced the notion of extremal rays and systematically used it in an attempt to construct minimal models in higher dimension, called the minimal model program. Danilov [2] introduced the notion of canonical and terminal singularities in conjunction with the factorization problem. This was developed by Reid [1][2][4] into a general theory of these singularities, which appear in an essential way in the minimal model program. The minimal model program is so far proven up to dimension 3 (See Mori [3] Kawamata [1][2][3] Kollar [1] Shokurov [1].) and for toric varieties in arbitrary dimension (See Reid [3]). In the steps of the minimal model program one is only allowed to contract a divisor into a variety with terminal singularities or to perform a flip, modifying some codimension 2 loci. This allows a factorization of a given projective birational morphism into such "elementary operations". An algorithm to factor birational maps among uniruled varieties, known as "Sarkisov's program", has also been developed and established in dimension 3 in general (See Sarkisov [1] Reid [5] Corti [1].) and in arbitrary dimension for toric varieties (See Matsuki [1]). It was hoped that the detailed understanding of such "elementary operations" would yield a solution, at least in dimension 3, to the classical factorization problem. Still, we do not know of a way to carry out this approach.

### **Resurgence of Toroidal Geometry**

It was a shock when De Jong [1] showed that the resolution of singularities in positive characteristics, though up to finite alterations, can be achieved and reduced to the toroidal case by an elegant application of the theory of the moduli space of (marked) stable curves. Resolution of singularities (easy but weak) in characteristic zero, without finite alteration, was also achieved shortly after by Abramovich-DeJong [1] and Bogomolov-Pantev [1] (See also Paranjape [1]), again reducing the general case to the toroidal case. When we saw that Abramovich-Karu [1] solved the problem of semi-stable reduction over higher dimensional base as an application of the method above, extending the original theorem by Kempf-Knudsen-Mumford-SaintDonat [1], the inspiration was developing into a belief that toric (toroidal) geometry is not just a baby version testing the real case but reflects the geometry in general and that the factorization problem can be approached in a similar manner by some reduction step yet to be found.

### **Local Version**

There is a local version of the factorization problem, formulated and proved in dimension 2 by Abhyankar [1]. Chritensen [1] posed the problem in general and solved it for some special cases in dimension 3. Here the varieties  $X_1$  and  $X_2$  for a birational map  $\phi : X_1 \dashrightarrow X_2$  are replaced by appropriate birational local rings dominated by a fixed valuation, and blowups are replaced by monoidal transforms subordinate to the valuation. This local conjecture was recently solved by S.D. Cutkosky in its strongest form in a series of papers (cf. Cutkosky [1][2][3]). The main ingredient of his proof is the "monomialization of a given birational map", i.e., the local version of the reduction step to the toroidal case. Thus the results of Cutkosky, though we do not use any of his method in our proof, not only gave us an psychological support that there is no local obstruction to solving the global

factorization conjecture but also presented another evidence supporting the belief mentioned above.

### Birational Cobordism and Connection with G.I.T.

Włodarczyk [2] provides the long-sought-after and key ingredient to reduce the factorization problem of GENERAL birational maps to that of TOROIDAL birational maps by revealing the true nature of Morelli's cobordism and hence opening the way to applying his combinatorial analysis to the general case via the theory of "birational cobordism", which can be regarded as an algebraic version of the Morse Theory. His theory can also be interpreted in the frame work of Geometric Invariant Theory and has a direct connection with the subject of the change of G.I.T. quotients associated to the change of linearizations, which has been intensively studied by Thaddeus [1][2] and others.

### Canonical Resolution of Singularities

There has been much progress after Hironaka's original work toward the understanding and simplification of the algorithms of resolution of singularities, notably by Bierstone-Milman [1] Villamayor [1] Encinas-Villamayor [1] among others. Since the elimination of points of indeterminacy, it has been a common consensus that the problem of resolution of singularities is in close connection with the factorization problem. In fact, the canonicity of these algorithms for resolution of singularities plays an indispensable role in our proof for the weak factorization.

### Solutions

The theory of birational cobordism by Włodarczyk [2] almost immediately factors a given birational map into locally toric birational transformations, the factorization which is a consequence of a local analysis in nature. But in order to apply Morelli's combinatorial algorithm, which is a global in nature, we still have to bridge the gap between "local" and "global".

Włodarczyk [3] provides this bridge by constructing a global combinatorial object, much like the conical complexes associated to toroidal embeddings in Kempf-Knudsen-Mumford-SaintDonat [1], and hence a solution for the weak factorization problem. Abramovich-Karu-Matsuki-Włodarczyk [1], meanwhile, provides this bridge by transforming the locally toric structure into a (global) toroidal structure by the process of torification. It is obtained by blowing up the "torific" ideal. The idea originates from the above-mentioned work of Abramovich-DeJong [1] for the weak resolution of singularities.

Both proofs share essentially the same difficulty toward the strong factorization conjecture, which remains as an open question.

### Today and Future

Though based upon the same belief, our proof presented here is different from what we had originally conceived as a possible approach to the factorization problem. King, in a joint paper with Abkulut, essentially presented what is now known as the toroidalization conjecture, our original approach, stated explicitly in Abramovich-Karu [1] Abramovich-Matsuki-Rashid [1].

**Toroidalization Conjecture.** *Let  $\phi : X_1 \dashrightarrow X_2$  be as above. Then there exist*

*sequences of blowups and blowdowns with smooth centers  $X'_1 \rightarrow X_1$  and  $X'_2 \rightarrow X_2$  such that the induced morphism  $\phi' : X'_1 \rightarrow X'_2$  becomes toroidal (with respect to suitably chosen toroidal structures on  $X'_1$  and  $X'_2$ ).*

The conjecture has been known to hold in dimension 2 by the original works of Abkulut-King [1], Abramovich-Karu [1] Karu [1] and that of Abramovich presented in Abramovich-Karu-Matsuki-Włodarczyk [1], as well as by a recent local algorithmic proof by Cutkosky-Piltant [1]. The toroidalization conjecture makes sense not only for a birational morphism but also for a morphism  $f : X \rightarrow Y$  between varieties of different dimensions, the flexibility which makes us hopeful for the inductional structure in a possible proof. (Cutkosky announced a proof for the case  $\dim X = 3$  and  $\dim Y = 2$  in the summer of 1999.)

An affirmative solution for the toroidalization conjecture will automatically provide one for the strong factorization conjecture.

Also by resolution of singularities, it is easily reduced to the following case of morphisms between toroidal embeddings: Let  $f : (U_X, X) \rightarrow (U_Y, Y)$  be a morphism between complete nonsingular toroidal embeddings with  $f : U_X = f^{-1}(U_Y) \rightarrow U_Y$  being smooth. Then there should exist sequences of blowups and blowdowns  $\sigma_X : (U_{X'}, X') \rightarrow (U_X, X)$  and  $\sigma_Y : (U_{Y'}, Y') \rightarrow (U_Y, Y)$  with smooth centers sitting in the boundaries so that the induced morphism  $f' : (U_{X'}, X') \rightarrow (U_{Y'}, Y')$  is “log-smooth”, or equivalently, toroidal. The notion of a morphism being log-smooth was introduced by Kato [1] together with that of the log category. (See also the log category and logarithmic ramification formula of Iitaka [1].) It gives us an intrinsic justification of the toroidalization conjecture, which otherwise might stand as a mere technical speculation. Interpreted in this way, the toroidalization conjecture may be regarded as the problem of resolution of singularities of morphisms (maps) in the log-category, similar to the problem of resolution of singularities of varieties in the usual category. The pursuit of a canonical algorithm for toroidalization is the future topic which invokes intensive research of today.

## CHAPTER 1. PRELIMINARIES

In the first part of this chapter, we recall some basic definitions and facts from Geometric Invariant Theory (G.I.T. for short, cf. Kirwan-Fogarty-Mumford [1]) and Toric Geometry (cf. Danilov [1] Fulton [1] Kempf-Knudsen-Mumford-SaintDonat [1] Oda [1]), mainly to fix the notation for these lecture notes.

The second part is devoted to the discussion of “**locally toric**” and “**toroidal**” structures. Since one of the critical issues in our argument is the distinction between the two notions and since in the existing literatures they appear in some mixed terminologies, the understanding of this part is crucial as well as basic.

In the third part of this chapter, we discuss the actions of the multiplicative group  $K^*$  of the base field on the locally and/or toroidal structures. In the course of studying birational transformations via birational cobordisms, we would like to introduce locally toric and/or toroidal structures not only on the birational cobordisms but also on their quotients by the actions of  $K^*$ . This naturally leads us to the (well-known) notion of a “strongly étale”  $K^*$ -equivariant morphism, utilized in the form of **Luna’s Fundamental Lemma** and **Luna’s Étale Slice Theorem**.

In these lecture notes, all the varieties are assumed to be irreducible and reduced schemes of finite type over an algebraically closed field  $K$  of characteristic zero, EXCEPT for Chapter 5 where we specifically try to loosen some of the restrictions.

### §1-1. Brief Review of G.I.T. and Toric Geometry

#### Review 1-1-1 (Geometric Invariant Theory).

Suppose a reductive group  $G$  acts on an algebraic variety  $X$ . We denote by  $X/G$  what we call the geometric quotient, i.e., the space of orbits, and by  $X//G$  the space of equivalence classes of orbits, where the equivalence relation is generated by the condition that two orbits are equivalent if their closures intersect.

These spaces are endowed with the structures (of algebraic varieties) which satisfy the usual universal property (if they exist in the category of algebraic varieties).

In most of our situations,

- $G$  is taken to be the multiplicative group  $K^*$  or a finite group,
- when we take the quotient  $X//G$ , a variety  $X$  is such that the closure of any orbit contains a unique closed orbit, i.e., contains a unique fixed point (See the definition of a quasi-elementary cobordism in Chapter 2),
- if a variety  $X$  is normal, then the quotient  $X/G$  or  $X//G$  is also a normal variety (if it exists in the category of algebraic varieties).

If  $X$  is endowed with an ample line bundle  $\mathcal{L}$  with a linear  $G$ -action compatible with the  $G$ -action on  $X$  (called a linearization of the action), then we denote by  $X_{\mathcal{L}}^{ss}$  the set of semi-stable points with respect to this linearization

$$X_{\mathcal{L}}^{ss} = \{x \in X; \exists s \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ for some } n \in \mathbb{N} \text{ s.t. } s(x) \neq 0\}$$

and by  $X_{\mathcal{L}}^{ss}//G$  the so-called G.I.T. quotient, which has the structure of a projective algebraic variety

$$X_{\mathcal{L}}^{ss}//G = \text{Proj } \oplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})^G.$$

We discuss more on G.I.T. related to the work of Thaddeus and others, i.e., in the subject of the change of the G.I.T. quotients associated to the change of linearizations in Chapter 2.

### Review 1-1-2 (Toric Geometry).

Let  $N \cong \mathbb{Z}^n$  be a lattice and  $\sigma \subset N_{\mathbb{R}} = N \otimes \mathbb{R}$  be a strictly convex rational polyhedral cone. We denote the dual lattice by  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  and the dual cone by  $\sigma^{\vee} \subset M_{\mathbb{R}} = M \otimes \mathbb{R}$ . The affine toric variety  $X$  associated to  $\sigma$  is defined to be

$$X = X(N, \sigma) = \text{Spec } K[M \cap \sigma^{\vee}],$$

containing the torus

$$X \supset T_X = \text{Spec } K[M].$$

The lattice  $N$  is identified with the group of one-parameter subgroups of  $T_X$  and the dual lattice with their characters. As we only consider toric varieties associated to the saturated lattices  $M \cap \sigma^{\vee}$  as above, they are all normal by definition.

More generally the toric variety associated to a fan  $\Sigma$  in  $N_{\mathbb{R}}$  is denoted by  $X(N, \Sigma)$ .

If  $X_1 = X(N, \Sigma_1)$  and  $X_2 = X(N, \Sigma_2)$  are toric varieties having the same lattice  $N$ , the embeddings of the torus  $T_{X_1} = T_{X_2} = \text{Spec } K[M]$  define an equivariant birational map  $\phi : X_1 \dashrightarrow X_2$ , called a toric birational map. This map  $\phi$  is a morphism if and only if every cone in  $\Sigma_1$  is contained in a cone in  $\Sigma_2$  and proper if and only if the support of  $\Sigma_1$  coincides with that of  $\Sigma_2$ .

Suppose that  $K^*$  acts on an affine toric variety  $X = X(N, \sigma)$  as a one-parameter subgroup of the torus  $T_X$ , corresponding to a lattice point  $a \in N$ . If  $t \in K^*$  and  $m \in M$ , the action on the monomial  $z^m$  is given by

$$t^*(z^m) = t^{(a, m)} \cdot z^m,$$

where  $(\cdot, \cdot)$  is the natural pairing on  $N \times M$ . (For  $m \in M$ , we denote its image in the semi-group algebra  $K[M \cap \sigma^{\vee}]$  by  $z^m$ .)

The  $K^*$ -invariant monomials correspond to the lattice points  $M \cap a^{\perp}$ , hence

$$X//K^* = \text{Spec } K[M \cap \sigma^{\vee} \cap a^{\perp}].$$

Let  $\pi : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/a \cdot \mathbb{R}$  be the projection. Then the lattice  $M \cap a^{\perp}$  is dual to the lattice  $\pi(N)$  and

$$M \cap \sigma^{\vee} \cap a^{\perp} = \text{Hom}_{\mathbb{Z}}(\pi(N), \mathbb{Z}) \cap \pi(\sigma)^{\vee}.$$

Therefore,  $X//K^* = X(\pi(N), \pi(\sigma))$  is again an affine toric variety defined by the lattice  $\pi(N)$  and the cone  $\pi(\sigma)$  (though  $\pi(\sigma)$  may no longer be strictly convex). This quotient is a geometric quotient precisely when  $\pi : \sigma \rightarrow \pi(\sigma)$  is a linear isomorphism (while the map between the lattices  $\pi : N \cap \sigma \rightarrow \pi(N) \cap \pi(\sigma)$  may not be an isomorphism).

#### §1-2. “Locally Toric” Structures vs. “Toroidal” Structures

There is some confusion in the existing literatures among the terminologies expressing the notion of toroidal embeddings and toroidal maps (cf. Kempf-Knudsen-Mumford-SaintDonat [1] Abramovich-Karu [2]) and that of toroidal varieties (cf.

Danilov [1] Włodarczyk [2]), the latter of which we prefer to call locally toric varieties (and locally toric maps). Since one of the critical issues in our argument is the distinction of the two notions “locally toric” and “toroidal”, we clarify our usage of these two terminologies below.

**Definition 1-2-1 (Locally Toric & Toroidal Structures).**

(i) A variety  $W$  is locally toric if for every closed point  $p \in W$  there exists a closed point  $x \in X$  in an affine toric variety such that we have an isomorphism of complete local rings over  $K$

$$\widehat{\mathcal{O}_{W,p}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X,x}}.$$

(ii) A pair  $(U_W, W)$  consisting of a dense open subset  $U_W \subset W$  contained in a variety  $W$  is called a toroidal embedding (or alternatively we say  $(U_W, W)$  has a toroidal structure) if for every closed point  $p \in W$  there exists a closed point  $x \in X$  in an affine toric variety such that we have an isomorphism of complete local rings over  $K$

$$\widehat{\mathcal{O}_{W,p}} \xrightarrow{\eta_1^*} \widehat{\mathcal{O}_{X,x}}$$

which induces an isomorphism of ideals defining the boundaries

$$\widehat{\mathcal{I}_{W-U_W}} = \mathcal{I}_{W-U_W} \otimes \widehat{\mathcal{O}_{W,p}} \xrightarrow{\eta_1^*} \widehat{\mathcal{I}_{X-T_X}} = \mathcal{I}_{X-T_X} \otimes \widehat{\mathcal{O}_{X,x}}$$

where  $\mathcal{I}_{W-U_W}$  (resp.  $\mathcal{I}_{X-T_X}$ ) is the ideal defining the reduced structure on the boundary  $W - U_W$  (resp.  $X - T_X$ ).

If all the irreducible components of the boundary  $W - U_W$  is normal, then  $(U_W, W)$  is called a toroidal embedding without self – intersection.

**Definition 1-2-2 (Locally Toric & Toroidal Birational Morphisms).**

(i) A proper birational morphism  $f : W_1 \rightarrow W_2$  between locally toric varieties is locally toric if for every closed point  $p_1 \in W_1$  mapping to  $p_2 = f(p_1) \in W_2$ , there exists a toric morphism  $\varphi : X_1 \rightarrow X_2$  mapping a closed point  $x_1 \in X_1$  to  $x_2 = \varphi(p_1) \in X_2$  such that we have a commutative diagram of homomorphisms between complete local rings

$$\begin{array}{ccc} \widehat{\mathcal{O}_{W_1,p_1}} & \xrightarrow{\eta_1^*} & \widehat{\mathcal{O}_{X_1,x_1}} \\ f^* \uparrow & & \uparrow \varphi^* \\ \widehat{\mathcal{O}_{W_2,p_2}} & \xrightarrow{\eta_2^*} & \widehat{\mathcal{O}_{X_2,x_2}}. \end{array}$$

(ii) A proper birational morphism  $f : (U_{W_1}, W_1) \rightarrow (U_{W_2}, W_2)$  between toroidal embeddings is toroidal if for every closed point  $p_1 \in W_1$  mapping to  $p_2 = f(p_1) \in W_2$ , there exists a toric morphism  $\varphi : X_1 \rightarrow X_2$  mapping a closed point  $x_1 \in X_1$  to  $x_2 = \varphi(p_1) \in X_2$  such that we have a commutative diagram of homomorphisms between complete local rings

$$\begin{array}{ccc} \widehat{\mathcal{O}_{W_1,p_1}} & \xrightarrow{\eta_1^*} & \widehat{\mathcal{O}_{X_1,x_1}} \\ f^* \uparrow & & \uparrow \varphi^* \\ \widehat{\mathcal{O}_{W_2,p_2}} & \xrightarrow{\eta_2^*} & \widehat{\mathcal{O}_{X_2,x_2}} \end{array}$$

which induces a commutative diagram of ideals defining the boundaries

$$\begin{array}{ccc} (\widehat{\mathcal{I}_{W_1 - U_{W_1}}})_{p_1} & \xrightarrow{\eta_1^*} & (\widehat{\mathcal{I}_{X_1 - U_{X_1}}})_{x_1} \\ f^* \uparrow & & \uparrow \varphi^* \\ (\widehat{\mathcal{I}_{W_1 - U_{W_1}}})_{p_2} & \xrightarrow{\eta_2^*} & (\widehat{\mathcal{I}_{X_1 - U_{X_1}}})_{x_2}. \end{array}$$

Recall that a birational map  $f : W_1 \dashrightarrow W_2$  is proper (cf. Iitaka [1]) if both projections  $p_1 : \Gamma_f \rightarrow W_1$  and  $p_2 : \Gamma_f \rightarrow W_2$  are proper morphisms in the usual sense, where  $\Gamma_f \subset W_1 \times W_2$  is the graph of  $f$ . (If  $f$  is well-defined as a morphism over a dense open subset  $U \subset W_1$ , then  $\Gamma_f$  is the closure in  $W_1 \times W_2$  of the graph of the morphism  $f|_U : U \rightarrow W_2$ . This closure is independent of the choice of the open subset  $U$ .)

### **Definition 1-2-3 (Locally Toric & Toroidal Birational Maps).**

(i) A proper birational map  $f : W_1 \dashrightarrow W_2$  between locally toric varieties is locally toric if there exists another locally toric variety  $Z$  which dominates both  $W_1$  and  $W_2$  by proper birational morphisms

$$W_1 \xleftarrow{f_1} Z \xrightarrow{f_2} W_2$$

where we require  $f_1$  and  $f_2$  to be locally toric as in Definition 1-2-2 (i).

(ii) A proper birational map  $f : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$  between toroidal embeddings is toroidal if there exists another toroidal embedding  $(U_Z, Z)$  which dominates both  $(U_{W_1}, W_1)$  and  $(U_{W_2}, W_2)$  by proper birational morphisms

$$(U_{W_1}, W_1) \xleftarrow{f_1} (U_Z, Z) \xrightarrow{f_2} (U_{W_2}, W_2)$$

where we require  $f_1$  and  $f_2$  to be toroidal as in Definition 1-2-2 (ii).

### **Remark 1-2-4.**

(i) Though we presented the notions of “locally toric” and “toroidal” as if they were parallel and of equal importance, the collection of locally toric varieties with locally toric birational morphisms and/or birational maps does NOT form a category, in clear contrast to the fact that the collection of toroidal embeddings with toroidal birational morphisms and/or toroidal birational maps does form a category as shown in Proposition 1-2-5.

A composition of locally toric birational morphisms or maps is NOT locally toric in general (cf. Proposition 1-2-5 (i) (ii)): We take  $g = f_2 \circ f_1$  to be a composite of the blowup  $f_2 : W_2 \rightarrow W_3$  of a point on a smooth threefold  $W_3$  and the blowup  $f_1 : W_1 \rightarrow W_2$  of a smooth curve on  $W_2$  which is not transversal to the exceptional divisor for  $f_2$ . Then while  $f_1$  and  $f_2$  are locally toric, the composition  $g$  is not.

It is NOT clear and probably not true (cf. Proposition 1-2-5 (iii)) that the definition of being locally toric for a birational morphism in Definition 1-2-2 (i) is compatible with the definition of being locally toric for a birational map in Definition 1-2-3 (i), though the author does not have a specific example showing incompatibility.

(ii) There are examples where a locally toric birational morphism  $f : W_1 \rightarrow W_2$  cannot be toroidal no matter how we introduce toroidal structures on  $W_1$  and  $W_2$ : Let  $W_2$  be a smooth surface. First we blow up a point on  $W_2$  and then blow up 3 or more distinct points on the exceptional divisor to obtain  $f : W_1 \rightarrow W_2$ .

**Proposition 1-2-5.**

- (i) A composition of two (proper and) toroidal birational morphisms is again (proper and) toroidal.
- (ii) A composition of two (proper and) toroidal birational maps is again (proper and) toroidal.
- (iii) A proper birational morphism between toroidal embeddings is toroidal in the sense of Definition 1-2-3 (ii) if and only if it is toroidal in the sense of Definition 1-2-2 (ii).

In particular, the collection of toroidal embeddings with toroidal birational morphisms and/or toroidal birational maps forms a category.

*Proof.*

The assertion (i) is proved in Abramovich-Karu [1] Karu [1] for a composition of general (not necessarily birational) toroidal morphisms.

The assertions (ii) and (iii) can be proved as an application of the following lemmas, after reducing the case of general toroidal embeddings to that of toroidal embeddings without self-intersection via some appropriate toroidal blowups and base changes.

**Lemma 1-2-6.** *Let  $(U_W, W)$  be a toroidal embedding. Then there exists a sequence  $(U_{\tilde{W}}, \tilde{W}) \rightarrow (U_W, W)$  of toroidal blowups with smooth centers (i.e., the centers analytically locally correspond to the smooth closures of orbits) such that the resulting toroidal embedding  $(U_{\tilde{W}}, \tilde{W})$  is nonsingular and without self-intersection.*

*Proof.*

First take the canonical resolution of singularities  $r : W^{res} \rightarrow W$  and set  $U_{W^{res}} = r^{-1}(U_W)$ . Then by the property ( $\spadesuit^{res} - 1$ ) of the canonical resolution (We refer the reader to Remark 4-1-1 for the details of the properties of the canonical resolution of singularities.) we conclude that  $r : (U_{W^{res}}, W^{res}) \rightarrow (U_W, W)$  is a toroidal morphism (between toroidal embeddings) obtained as a sequence of toroidal blowups with smooth centers. Now starting from blowing up the 0-dimensional strata of the boundary components of  $(U_{W^{res}}, W^{res})$ , we consecutively blow up (the closures of) the strict transforms of the original  $k$ -dimensional strata of the boundary of the original toroidal structure  $(U_{W^{res}}, W^{res})$  for  $k = 1, \dots, \dim W - 1$  to obtain  $s : (U_{\tilde{W}} = s^{-1}(U_{W^{res}}), \tilde{W}) \rightarrow (U_{W^{res}}, W^{res})$ . (Note that the closures of the strata of the boundary components are the closures of the generic points which, analytically locally, correspond to the intersections of the boundary divisors of the toroidal charts.) This is a sequence of toroidal blowups with smooth centers so that the resulting toroidal embedding  $(U_{\tilde{W}}, \tilde{W})$  is nonsingular and without self-intersection.

**Lemma 1-2-7.** *Let  $f : (U_{W_1}, W_1) \rightarrow (U_{W_2}, W_2)$  be a proper and toroidal birational morphism between toroidal embeddings without self-intersection. Then  $f$  is allowable (in the sense of Kempf-Knudsen-Mumford-SaintDonat [1]). In particular, such proper and toroidal birational morphisms over  $(U_{W_2}, W_2)$  are in one-to-one correspondence with the subdivisions of the conical complex associated to the toroidal embedding  $(U_{W_2}, W_2)$  without self-intersection.*

*Proof.*

For a proof, we refer the reader to Abramovich-Karu [1] Karu [1] and Kempf-Knudsen-Mumford-SaintDonat [1].

In order to see the assertion (ii), suppose

$$f : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2) \text{ and } g : (U_{W_2}, W_2) \dashrightarrow (U_{W_3}, W_3)$$

are proper and toroidal birational maps among toroidal embeddings. Let

$$(U_{W_1}, W_1) \xleftarrow{f_1} (U_Z, Z) \xrightarrow{f_2} (U_{W_2}, W_2) \text{ and } (U_{W_2}, W_2) \xleftarrow{g_2} (U_{Z'}, Z') \xrightarrow{g_3} (U_{W_3}, W_3)$$

be some toroidal embeddings dominating  $(U_{W_1}, W_1)$ ,  $(U_{W_2}, W_2)$  and  $(U_{W_3}, W_3)$  by proper and toroidal birational morphisms as required in Definition 1-2-3 (ii). By taking a sequence of toroidal blowups with smooth centers of  $(U_{W_2}, W_2)$  using Lemma 1-2-6, we can make it without self-intersection. Since  $f_2$  and  $g_2$  are toroidal, we can “pull-back” these toroidal blowups to the toroidal blowups on  $(U_Z, Z)$  and on  $(U_{Z'}, Z')$ . Again using Lemma 1-2-6 and the assertion (i), we may assume that  $(U_Z, Z)$ ,  $(U_{W_2}, W_2)$  and  $(U_{Z'}, Z')$  are all without self-intersection. By Lemma 1-2-7, the morphisms  $f_2$  and  $g_3$  are associated with the subdivisions  $\Delta_Z$  and  $\Delta_{Z'}$  of the conical complex  $\Delta_{W_2}$ . Let  $(U_{\hat{Z}}, \hat{Z}) \rightarrow (U_{W_2}, W_2)$  be the toroidal embedding associated to the common refinement  $\Delta_{\hat{Z}}$  of  $\Delta_Z$  and  $\Delta_{Z'}$ . Then by construction  $(U_{\hat{Z}}, \hat{Z})$  dominates  $(U_Z, Z)$  and  $(U_{Z'}, Z')$  by proper and toroidal birational morphisms

$$(U_{W_1}, W_1) \xleftarrow{f_1} (U_Z, Z) \xleftarrow{\hat{f}_1} (U_{\hat{Z}}, \hat{Z}) \xrightarrow{\hat{g}_3} (U_{Z'}, Z') \xrightarrow{g_3} (U_{W_3}, W_3).$$

Since the compositions  $f_1 \circ \hat{f}_1$  and  $g_3 \circ \hat{g}_3$  are both toroidal by Lemma 1-2-6, we conclude that the composition  $g \circ f$  is also a toroidal birational map by Definition 1-2-3 (ii).

In order to see the assertion (iii), let  $f : (U_{W_1}, W_1) \rightarrow (U_{W_2}, W_2)$  be a proper birational morphism between toroidal embeddings. If  $f$  is toroidal as a birational morphism according to Definition 1-2-2 (ii), then it is obviously toroidal as a birational map according to Definition 1-2-3 (ii) by taking the diagram

$$(U_{W_1}, W_1) \xleftarrow{\text{identity}} (U_{W_1}, W_1) \xrightarrow{f} (U_{W_2}, W_2).$$

Thus we only have to show the converse, i.e., if  $f$  is toroidal as a birational map as in Definition 1-2-3 (ii), then it is toroidal as a birational morphism as in Definition 1-2-2 (ii).

According to Definition 1-2-3 (ii), there exists another toroidal embedding  $(U_Z, Z)$  which dominates both  $(U_{W_1}, W_1)$  and  $(U_{W_2}, W_2)$  by proper birational morphisms

$$(U_{W_1}, W_1) \xleftarrow{f_1} (U_Z, Z) \xrightarrow{f_2} (U_{W_2}, W_2)$$

where  $f_1$  and  $f_2$  are proper and toroidal birational morphisms according to Definition 1-2-2 (ii). By Lemma 1-2-6 and Proposition 1-2-5 (i), we may assume that  $(U_Z, Z)$  is a nonsingular toroidal embedding without self-intersection. We denote by  $E_k$  the irreducible components of the boundary  $Z - U_Z$ .

Let  $p_1 \in W_1$  be a closed point mapping to  $p_2 = f(x_1) \in W_2$ .

Via the analytic isomorphisms as in the definition of the toroidal structures

$$\begin{aligned}\widehat{\mathcal{O}_{W_1,p_1}} &\xrightarrow{\sim} \widehat{\mathcal{O}_{X_1,x_1}} \\ \widehat{\mathcal{O}_{W_2,p_2}} &\xrightarrow{\sim} \widehat{\mathcal{O}_{X_2,x_2}}\end{aligned}$$

we pull back the standard coordinates for the tori

$$\begin{aligned}\{z_1, \dots, z_n\} \text{ for the torus } T_{X_1} = \text{Spec } K[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \\ \{w_1, \dots, w_n\} \text{ for the torus } T_{X_2} = \text{Spec } K[w_1^{\pm 1}, \dots, w_n^{\pm 1}],\end{aligned}$$

which we denote by the same letters by abuse of notation.

We only have to show the claim that there exists a unimodular  $n \times n$  matrix  $A = \{a_{ij}\}$  and units  $u_i \in \widehat{\mathcal{O}_{W_1,p_1}}$  for  $i, j = 1, \dots, n$  such that

$$f^* w_i = u_i \prod z_j^{a_{ij}} \text{ in the field of fractions of } \widehat{\mathcal{O}_{W_1,p_1}}.$$

Remark that

$$\begin{aligned}\widehat{f}_1 : (U_Z, Z) \times \text{Spec } \widehat{\mathcal{O}_{W_1,p_1}} &\rightarrow (U_{W_1}, W_1) \times \text{Spec } \widehat{\mathcal{O}_{W_1,p_1}} \\ \widehat{f}_2 : (U_Z, Z) \times \text{Spec } \widehat{\mathcal{O}_{W_2,p_2}} &\rightarrow (U_{W_2}, W_2) \times \text{Spec } \widehat{\mathcal{O}_{W_2,p_2}}\end{aligned}$$

are both proper and birational toroidal morphisms between toroidal embeddings WITHOUT self-intersection and hence allowable and that they correspond to the subdivisions of the conical complexes by Lemma 1-2-7, which we denote by

$$\begin{aligned}\Delta_{\widehat{f}_1} : \Delta_{Z,p_1} &\rightarrow \Delta_{W_1,p_1} \\ \Delta_{\widehat{f}_2} : \Delta_{Z,p_2} &\rightarrow \Delta_{W_2,p_2}.\end{aligned}$$

Remark also that  $\Delta_{Z,p_1}$  can be considered as a subcomplex of  $\Delta_{Z,p_2}$ , since  $f_1^{-1}(p_1) \subset f_2^{-1}(p_2)$ .

Suppose  $\dim \Delta_{Z,p_1} = \dim \Delta_{Z,p_2}$  (while in general we have  $\dim \Delta_{Z,p_1} \leq \dim \Delta_{Z,p_2}$ ). Let  $d_1$  be the minimum among the dimensions of the strata whose closures are the connected components of the intersections of the  $E_k$  and which have nonempty intersection with  $f_1^{-1}(p_1)$ . Let  $d_2$  be the minimum among the dimensions of the strata whose closures are the connected components of the intersections of the  $E_k$  and which have nonempty intersection with  $f_2^{-1}(p_2)$ . Then the assumption is equivalent to  $d_1 = d_2$  (while in general we have  $d_1 \geq d_2$ ). In this case,  $\Delta_{Z,p_1}$  is a conical complex of full dimension embedded in another conical complex  $\Delta_{Z,p_2}$  with the compatible integral structures. More precisely, since  $f_2$  factors as  $f_2 = f_1 \circ f$ , we can take a  $\mathbb{Z}$ -basis of the integral structure for  $\Delta_{W_1,p_1}$  (which hence provides a common  $\mathbb{Z}$ -basis for each of the cones of the maximal dimension in  $\Delta_{Z,p_1}$ ) which is compatible with a  $\mathbb{Z}$ -basis of the integral structure of  $\Delta_{W_2,p_2}$  (which hence provides a common  $\mathbb{Z}$ -basis for each of the cones of the maximal dimension in  $\Delta_{Z,p_2}$ ). It follows that there exists a unimodular matrix  $A = \{a_{ij}\}$  such that for all  $z \in f_1^{-1}(p_1) \subset Z$  we have

$$f_2^* w_i = (u_i)_z \prod z_j^{a_{ij}} \text{ in the field of fractions of } \widehat{\mathcal{O}_{Z,z}}$$

for some units  $(u_i)_z \in \widehat{\mathcal{O}_{Z,z}}$  for  $i, j = 1, \dots, n$ . Since  $f_1 : Z \rightarrow W_1$  is proper and  $z \in f_1^{-1}(p_1)$  is arbitrary, we have the claim.

Suppose  $\dim \Delta_{Z,p_1} < \dim \Delta_{Z,p_2}$ , i.e.,  $d_1 > d_2$ . Let  $S$  be a stratum of dimension  $\dim S = d_1$  with nonempty intersection with  $f_1^{-1}(p_1)$ . Then there exists another stratum  $S' \subset S$  of dimension  $\dim S' = d_2$  with nonempty intersection with  $f_2^{-1}(p_2)$ . Take a point  $z' \in S' \cap f_2^{-1}(p_2)$  and let  $p'_1 = f_1(z')$ . By the argument for the previous case, for the pull-backs via the isomorphism as in the definition of the toroidal structure

$$\widehat{\mathcal{O}_{W_1,p'_1}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X'_1,x'_1}}$$

of the standard coordinates

$$\{z'_1, \dots, z'_n\} \text{ for the torus } T_{X'_1} = \text{Spec } K[z'_1^{\pm 1}, \dots, z'_n^{\pm 1}]$$

there exists a unimodular matrix  $A = \{a_{ij}\}$  such that

$$f_2^* w_i = (u_i)' \prod (z'_j)^{a_{ij}}$$

for some units  $(u_i)' \in \widehat{\mathcal{O}_{W_1,p'_1}}$ .

Note that every divisor  $E_k$  with  $E_k \cap f_1^{-1}(p_1) \neq \emptyset$  we have  $E_k \cap f_1^{-1}(p'_1)$  by the minimality of the dimension  $d_1 = \dim S$  of the stratum  $S$  and by the allowability of the morphism

$$\widehat{f}_1 : (U_Z, Z) \times \text{Spec } \widehat{\mathcal{O}_{W_1,p_1}} \rightarrow (U_{W_1}, W_1) \times \text{Spec } \widehat{\mathcal{O}_{W_1,p_1}}.$$

These also imply that there exists an isomorphism as in the definition of the toroidal structure

$$\widehat{\mathcal{O}_{W_1,p_1}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X_1,x_1}}$$

which pulls back the standard coordinates

$$\{z_1, \dots, z_n\} \text{ for the torus } T_{X_1} = \text{Spec } K[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$$

with the property

$$\text{ord}_{E_k}(z_j) = \text{ord}_{E_k}(z'_j)$$

for all  $E_k$  with  $E_k \cap f_1^{-1}(p_1)$  and  $j = 1, \dots, n$ . (Roughly put, we can find a point  $p''_1$  which sits in the same stratum as  $p_1$  with respect to the natural stratification of the boundary  $W_1 - U_{W_1}$  and hence the toroidal structures at  $p_1$  and  $p''_1$  are isomorphic, and which is close to  $p'_1$  and hence admits the same locally toric charts as  $p'_1$ .) Therefore, we have again for an arbitrary point  $z \in f_1^{-1}(p_1)$

$$f_2^* w_i = (u_i)_z \prod z_j^{a_{ij}} \text{ in the field of fractions of } \widehat{\mathcal{O}_{Z,z}}$$

for some units  $(u_i)_z \in \widehat{\mathcal{O}_{Z,z}}$  for  $i, j = 1, \dots, n$ . Since  $f_1 : Z \rightarrow W_1$  is proper, we have the claim.

This completes the proof of Proposition 1-2-5.

**Remark 1-2-8.**

(i) As we have seen in Remark 1-2-4 (i), a composition of two locally toric birational morphisms is not necessarily locally toric, since the locally toric structures for the two locally toric morphisms may not be compatible in the middle. Even worse, the locally toric structures for the morphisms  $W_1 \xleftarrow{f_1} Z$  and  $Z \xrightarrow{f_2} W_2$  in Definition 1-2-3 (i) of a locally toric birational map may not be compatible on  $Z$ . Thus it is nontrivial to prove the assertions (i), (ii) and (iii) in Proposition 1-2-5, settling a rather subtle issue of the compatibility of toroidal structures for several different toroidal morphisms.

(ii) We define a stronger version of locally toric and toroidal birational maps, namely V-locally toric and V-toroidal birational maps below. These turn out to be the only maps we have to consider in our note, the fact which enables us to avoid the subtlety as above.

Therefore, except for Lemma 1-2-6 and Lemma 1-2-7, we will NOT use the results of Proposition 1-2-5 in the later argument of our proof.

Actually the only place where Lemma 1-2-7 appears in our note is when we use the allowability in the proof of Theorem 4-2-1. In the case of V-toroidal birational maps the allowability (after reducing the factorization to that of a birational map between toroidal embeddings without self-intersection) in Theorem 4-2-1 is automatic and hence our argument does not use Lemma 1-2-7, though it appears in Theorem 4-2-1 dealing with the factorization of general toroidal birational maps.

**Remark 1-2-9 (Comparison with the definitions in Abramovich-Karu-Matsuki-Włodarczyk [1]).**

In Abramovich-Karu-Matsuki-Włodarczyk [1], they give more restrictive definitions for locally toric and toroidal structures and locally toric and toroidal birational morphisms/maps via the use of (allowable) Zariski locally toric and toroidal charts.

Definition of locally toric and toroidal structures in Abramovich-Karu-Matsuki-Włodarczyk [1]:

(i) A variety  $W$  is locally toric if for every closed point  $p \in W$  there exists a Zariski open neighborhood  $V_p \subset W$  of  $p$  and an étale morphism  $\eta_p : V_p \rightarrow X_p$  to a toric variety  $X_p$ .

(ii) A pair  $(U_W, W)$  consisting of a dense open subset  $U_W \subset W$  in a variety  $W$  is called a toroidal embedding if for every closed point  $p \in W$  there exists a Zariski open neighborhood  $V_p$  and an étale morphism  $\eta_p : V_p \rightarrow X_p$  to a toric variety such that  $\eta_p^{-1}(T_{X_p}) = U_W$  where  $T_{X_p}$  is the torus for  $X_p$ .

Definition of locally toric and toroidal morphisms in Abramovich-Karu-Matsuki-Włodarczyk [1]:

(i) A proper birational morphism  $f : W_1 \rightarrow W_2$  between locally toric varieties is called locally toric if for every closed point  $p_2 \in W_2$  there exists a Zariski open neighborhood  $V_{p_2}$  and an étale morphism  $\eta_{p_2} : V_{p_2} \rightarrow X_{p_2}$  to a toric variety, together

with a toric proper birational morphism  $Y_{p_2} \rightarrow X_{p_2}$  such that we have a fiber square

$$\begin{array}{ccc} f^{-1}(V_{p_2}) = V_{p_2} \times_{X_{p_2}} Y_{p_2} & \rightarrow & Y_{p_2} \\ \downarrow & \square & \downarrow \\ V_{p_2} & \rightarrow & X_{p_2} \end{array}$$

(ii) A proper birational morphism  $f : (U_{W_1}, W_1) \rightarrow (U_{W_2}, W_2)$  between toroidal embeddings is called toroidal if for every closed point  $p_2 \in W_2$  there exists a Zariski open neighborhood  $V_{p_2}$  and an étale morphism  $\eta_{p_2} : V_{p_2} \rightarrow X_{p_2}$  to a toric variety, together with a toric proper birational morphism  $Y_{p_2} \rightarrow X_{p_2}$  such that we have a fiber square of toroidal embeddings

$$\begin{array}{ccc} (U_{W_1} \cap f^{-1}(V_{p_2}), f^{-1}(V_{p_2})) = V_{p_2} \times_{X_{p_2}} (T_{Y_{p_2}}, Y_{p_2}) & \rightarrow & (T_{Y_{p_2}}, Y_{p_2}) \\ \downarrow & \square & \downarrow \\ (U_{W_2} \cap V_{p_2}, V_{p_2}) = V_{p_2} \times_{X_{p_2}} (T_{X_{p_2}}, X_{p_2}) & \rightarrow & (T_{X_{p_2}}, X_{p_2}) \end{array}$$

Definition of locally toric and toroidal birational maps in Abramovich-Karu-Matsuki-Włodarczyk [1]:

(i) A proper birational map  $f : W_1 \dashrightarrow W_2$  between locally toric varieties is locally toric if there exists another locally toric variety  $Z$  which dominates both  $W_1$  and  $W_2$  by proper birational morphisms

$$W_1 \xleftarrow{f_1} Z \xrightarrow{f_2} W_2$$

where we require  $f_1$  and  $f_2$  to be locally toric as in (i) above.

(ii) A proper birational map  $f : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$  between toroidal embeddings is toroidal if there exists another toroidal embedding  $(U_Z, Z)$  which dominates both  $(U_{W_1}, W_1)$  and  $(U_{W_2}, W_2)$  by proper birational morphisms

$$(U_{W_1}, W_1) \xleftarrow{f_1} (U_Z, Z) \xrightarrow{f_2} (U_{W_2}, W_2)$$

where we require  $f_1$  and  $f_2$  to be toroidal as in (ii) above.

The above definitions differ from ours in several subtle ways. Here are a couple of differences:

- The toroidal embeddings in their sense are automatically without self-intersection, while those in our sense may not. For example, a pair  $(U_S, S)$  consisting of a nonsingular surface  $S$  with an irreducible nodal curve as a boundary component  $S - U_S$  is a toroidal embedding in our sense but not in their sense.

- Locally toric birational morphisms in our sense may not be locally toric in their sense. For example, as in Remark 1-2-4, if we blowup a point on a smooth surface  $W_2$  and further blowup three or more points on the exceptional divisor to obtain  $W_1$ , then the morphism  $f : W_1 \rightarrow W_2$  is locally toric in our sense but not in their sense.

On the other hand, we would like to emphasize that for the category of toroidal embeddings WITHOUT self-intersection and proper birational morphisms and maps, their definitions and our definitions coincide.

In fact, we have the following:

- The toroidal embedding without self-intersection in our sense always has a Zariski toroidal chart as required in their definition.

Let  $p \in (U_W, W)$  be a closed point in a toroidal embedding without self-intersection in our sense and let  $x \in X$  be a closed point in an affine toric variety as described in Definition 1-2-1 (ii). The Cartier divisors in  $\text{Spec } \widehat{\mathcal{O}_{W,p}} = \text{Spec } \widehat{\mathcal{O}_{X,x}}$  supported on the support of  $\widehat{\mathcal{O}_{W,p}/I_{W-U_W}} = \widehat{\mathcal{O}_{X,x}/I_{X-T_X}}$  give rise to a lattice in a cone of effective Cartier divisors (the dual cone sits in the conical complex associated to  $(U_W, W)$ , which is well-defined as  $(U_W, W)$  is without self-intersection). We only provide a proof in case the dual cone has the full dimension (being equal to  $n = \dim W$ ), leaving the general case as an exercise to the reader. Let  $z_1, \dots, z_n$  be the standard coordinates for the torus  $T_X = \text{Spec } K[z_1^\pm, \dots, z_n^\pm]$ . Since  $z_1, \dots, z_n$  define Cartier divisors which form a  $\mathbb{Z}$ -basis of the lattice and since  $(U_W, W)$  is without self-intersection, we can find rational functions  $r_1, \dots, r_n \in K(W)$  for  $W$  such that the Cartier divisors associated to them coincide, analytically locally at  $p$ , with those associated to  $z_1, \dots, z_n$ . Now let  $m_1, \dots, m_l$  be the monomials in  $z_1, \dots, z_n$  generating the affine coordinate ring for  $X$  and hence the maximal ideal corresponding to the point  $x \in X$ , and let  $m'_1, \dots, m'_l$  be the same monomials but in  $r_1, \dots, r_l$ . In a suitable affine Zariski open neighborhood  $V_p$ , we may assume all the monomials  $m'_1, \dots, m'_l$  are regular over  $V_p$ . It is easy to see  $r_1, \dots, r_n$  are transcendental over  $K$ , as well as  $z_1, \dots, z_n$  are over  $K$ . Therefore, we have a well-defined morphism

$$\eta_p : V_p \rightarrow X_p = X$$

defined by sending  $m_1, \dots, m_l$  to  $m'_1, \dots, m'_l$ , respectively. Now by construction we see that the homomorphism

$$\eta_p^* : \widehat{\mathcal{O}_{X,x}} \rightarrow \widehat{\mathcal{O}_{V,p}}$$

is surjective and that it is also injective as so is  $\eta_p^* : A(X) \rightarrow A(V_p)$ , hence an isomorphism. (Note that we are in the situation where we DO have the preservation of injectivity at the completion level, unlike the one in Gabrielov [1][2]’s counterexample to a conjecture of Grothendieck [1]. See, e.g., Hübl [1].) Therefore, by shrinking  $V_p$ , we may assume  $\eta_p$  is étale. By shrinking  $V_p$  further, we may also assume that all the irreducible divisors in  $V_p \cap (W - U_W)$  contain  $p$  and hence  $\eta_p^{-1}(T_X) = U_W \cap V_p$ . Thus  $\eta_p : V_p \rightarrow X_p$  is a desired Zariski toroidal chart.

- The toroidal birational morphisms in our sense are toroidal in their sense.

Let  $f : (U_{W_1}, W_1) \rightarrow (U_{W_2}, W_2)$  be a proper birational morphism which is toroidal in our sense according to Definition 1-2-2 (ii). For a point  $p_2 \in W_2$  we take a Zariski toroidal chart  $\eta_p : V_{p_2} \rightarrow X_{p_2}$  constructed as above. By shrinking  $X_{p_2}$  we may assume all the orbits in  $X_{p_2}$  have nonempty intersection with the image  $\eta_p(V_{p_2})$ . Then it is easy to see that the Cartier divisors in  $V_{p_2}$  supported on  $V_{p_2} \cap (W - U_W)$  are in one-to-one correspondence with Cartier divisors in  $X_{p_2}$  supported on  $X_{p_2} - T_{X_{p_2}}$  and that the conical complexes  $\Delta_{V_{p_2}}$  associated to  $V_{p_2} \cap (W - U_W)$  and  $\Delta_{X_{p_2}}$  associated to  $X_{p_2}$  coincide. Now by the results of Abramovich-Karu [1] Karu [1],  $f$  is allowable and hence corresponds to a subdivision of  $\Delta_{V_{p_2}}$  over  $V_{p_2}$ . By taking  $Y_{p_2} \rightarrow X_{p_2}$  to be the toric birational morphism corresponding to the same

subdivision of  $\Delta_{X_{p_2}}$ , we obtain the commutative diagram of fiber squares

$$\begin{array}{ccc} (U_{W_1} \cap f^{-1}(V_{p_2}), f^{-1}(V_{p_2})) = V_{p_2} \times_{X_{p_2}} (T_{Y_{p_2}}, Y_{p_2}) & \rightarrow & (T_{Y_{p_2}}, Y_{p_2}) \\ \downarrow & \square & \downarrow \\ (U_{W_2} \cap V_{p_2}, V_{p_2}) = V_{p_2} \times_{X_{p_2}} (T_{X_{p_2}}, X_{p_2}) & \rightarrow & (T_{X_{p_2}}, X_{p_2}). \end{array}$$

Therefore,  $f$  is also toroidal in their sense.

Now that we see that the their definitions and our definitons coincide for the category of toroidal embeddings without self-intersection, the statements of Proposition 1-2-5 (i), (ii) and (iii) also hold using their definitions in the category of toroidal embeddings without self-intersection.

#### Definition 1-2-10.

(i) A proper birational map  $f : W_1 \dashrightarrow W_2$  between locally toric varieties is called  $V$ -locally toric if there exists another locally toric variety  $Y$  which both  $W_1$  and  $W_2$  dominate by proper birational morphisms

$$W_1 \rightarrow Y \leftarrow W_2$$

such that for every closed point  $y \in Y$  we can find étale morphisms  $i : V \rightarrow Y$  with  $y \in i(V)$  and  $\eta : V \rightarrow X_Y$  to an affine toric variety  $X_Y$  which some toric varieties  $X_1$  and  $X_2$  dominate by proper toric birational morphisms

$$X_1 \rightarrow X_Y \leftarrow X_2$$

with the property that when we take the fiber-products with  $V$  they coincide, i.e., we have a commutative diagram of the form

$$\begin{array}{ccccc} W_1 & \rightarrow & Y & \leftarrow & W_2 \\ \uparrow & \square & \uparrow & \square & \uparrow \\ W_1 \times_Y V & \rightarrow & V & \leftarrow & W_2 \times_Y V \\ \parallel & & \parallel & & \parallel \\ X_1 \times_{X_Y} V & \rightarrow & V & \leftarrow & X_2 \times_{X_Y} V \\ \downarrow & \square & \downarrow & \square & \downarrow \\ X_1 & \rightarrow & X_Y & \leftarrow & X_2. \end{array}$$

(Note that  $X_Y$  and hence  $X_1$  &  $X_2$  are allowed to vary depending on the points  $y \in Y$ .)

(ii) A proper birational map  $f : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$  between toroidal embeddings is called  $V$ -toroidal if there exists another toroidal embedding  $(U_Y, Y)$  which both  $(U_{W_1}, W_1)$  and  $(U_{W_2}, W_2)$  dominate by proper birational morphisms

$$(U_{W_1}, W_1) \rightarrow (U_Y, Y) \leftarrow (U_{W_2}, W_2)$$

such that for every closed point  $y \in Y$  we can find étale morphisms  $i : V \rightarrow Y$  with  $y \in i(V)$  and  $\eta : V \rightarrow X_Y$  to an affine toric variety  $X_Y$  which some toric varieties  $X_1$  and  $X_2$  dominate by proper toric birational morphisms

$$X_1 \rightarrow X_Y \leftarrow X_2$$

with the property that when we take the fiber-products with  $V$  they coincide, i.e., we have a commutative diagram of the form

$$\begin{array}{ccccc} (U_{W_1}, W_1) & \rightarrow & (U_Y, Y) & \leftarrow & (U_{W_2}, W_2) \\ \uparrow & \square & \uparrow & \square & \uparrow \\ (U_{W_1}, W_1) \times_Y V & \rightarrow & (U_Y, Y) \times_Y V & \leftarrow & (U_{W_2}, W_2) \times_Y V \\ \parallel & & \parallel & & \parallel \\ (T_{X_1}, X_1) \times_{X_Y} V & \rightarrow & (T_{X_Y}, X_Y) \times_Y V & \leftarrow & (U_{X_2}, X_2) \times_{X_Y} V \\ \downarrow & \square & \downarrow & \square & \downarrow \\ (T_{X_1}, X_1) & \rightarrow & (T_{X_Y}, X_Y) & \leftarrow & (T_{X_2}, X_2). \end{array}$$

(Note again that  $X_Y$  and hence  $X_1$  &  $X_2$  are allowed to vary depending on the points  $y \in Y$ .)

### §1-3. $K^*$ -action on locally toric and toroidal structures

**Definition 1-3-1 (Strongly Étale).** Let  $V$  and  $X$  be affine varieties with  $K^*$ -actions and let

$$\eta : V \rightarrow X$$

be a  $K^*$ -equivariant étale morphism. Then  $\eta$  is said to be **strongly étale** if

(i) the quotient map

$$\eta//K^* : V//K^* \rightarrow X//K^*$$

is étale, and

(ii) the natural map

$$V \rightarrow X \times_{X//K^*} V//K^*$$

is an isomorphism.

**Remark 1-3-2.**

(i) The statement below does NOT hold in general:

For a  $K^*$ -equivariant étale morphism  $\eta : V \rightarrow X$  between affine varieties with  $K^*$ -actions, the quotient map  $\eta//K^* : V//K^* \rightarrow X//K^*$  is also étale.

For example, take

$$V = \text{Spec } K[u, u^{-1}, y] \text{ with the } K^*\text{-action given by } t \cdot (u, y) = (tu, t^{-1}y)$$

and

$$X = \text{Spec } K[w, w^{-1}, y] \text{ with the } K^*\text{-action given by } t \cdot (w, y) = (t^2w, t^{-1}y),$$

while a  $K^*$ -equivariant étale morphism  $\eta : V \rightarrow X$  is associated to the ring homomorphism  $\eta^* : K[w, y] \rightarrow K[u, y]$  defined by  $\eta^*(w) = u^2$ . Then the quotient map is given by

$$\eta//K^* : V//K^* = \text{Spec } K[uy] \rightarrow X//K^* = \text{Spec } K[wy^2]$$

which ramifies over the origin as  $wy^2 = (uy)^2$ .

The “reason” why  $\eta//K^*$  fails to be étale is that in general the stabilizer  $\text{Stab}(v) \subset K^*$  of a point  $v \in V$  is “smaller” than the stabilizer  $\text{Stab}(x) \subset K^*$  of the image  $x = \eta(v)$ . (The inclusion  $\text{Stab}(v) \subset \text{Stab}(x) \subset K^*$  is obvious.) Thus, roughly speaking, the quotient  $X//K^*$  is divided more than  $V//K^*$  by the difference  $\text{Stab}(x)/\text{Stab}(v)$ . In the above example, if a point  $p \in V$  has the coordinate  $y = 0$  the stabilizer is  $\text{Stab}(v) = \{1\} \subset K^*$ , while the image  $x = \eta(v)$  has the stabilizer  $\text{Stab}(x) = \{\pm 1\} \subset K^*$ .

It is not difficult to come up with an example where  $V//K^*$  is nonsingular while  $X//K^*$  is singular, and hence  $\eta//K^* : V//K^* \rightarrow X//K^*$  obviously fails to be étale.

(ii) On the other hand, if  $\eta : V \rightarrow X$  is strongly étale between affine varieties with  $K^*$ -actions, it follows from the condition (ii) that

$$\text{Stab}(v) = \text{Stab}(x) \quad \forall v \in V \text{ where } x = \eta(v).$$

(iii) Luna’s Fundamental Lemma stated below can be considered as the statement in (i) with the extra condition that  $\text{Stab}(v) = \text{Stab}(x)$  for all  $v \in V$  where  $x = \eta(v)$  is the image, when properly interpreted. Therefore, in a very rough sense, a  $K^*$ -equivariant étale morphism  $\eta : V \rightarrow X$  is strongly étale if and only if the stabilizers are preserved by  $\eta$ .

### Definition 1-3-3 (Locally Toric and Toroidal $K^*$ -actions).

(i) Let  $W$  be a locally toric variety with a  $K^*$ -action. We say that the action is locally toric if for every closed point  $p \in W$  we can find a  $K^*$ -invariant affine neighborhood  $p \in U_p \subset W$ , an affine variety  $V_p$  with a  $K^*$ -action and an affine toric variety  $X_p$  with  $K^*$  acting as a one-parameter subgroup such that we have  $K^*$ -equivariant and strongly étale morphisms  $\eta_p$  and  $i_p$

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p.$$

We call such a set of strongly étale morphisms **Luna’s locally toric chart at p** for the locally toric  $K^*$ -action.

(ii) Let  $(U_W, W)$  be a toroidal embedding with a  $K^*$ -action. We say that the action is toroidal if for every closed point  $p \in W$  we can find a  $K^*$ -invariant affine neighborhood  $p \in U_p \subset W$ , an affine toroidal embedding  $(U_{V_p}, V_p)$  with a  $K^*$ -action and an affine toric variety  $(T_{X_p}, X_p)$  with  $K^*$  acting as a one-parameter subgroup such that we have  $K^*$ -equivariant and strongly étale morphisms  $\eta_p$  and  $i_p$

$$(T_{X_p}, X_p) \xleftarrow{\eta_p} (U_{V_p}, V_p) \xrightarrow{i_p} (U_W \cap U_p, U_p),$$

where in the condition (ii) of  $\eta_p$  and  $i_p$  being strongly étale as stated in Definition 1-3-1 we require that the natural morphisms

$$\begin{aligned} (U_{V_p}, V_p) &\rightarrow (T_{X_p}, X_p) \times_{X_p//K^*} V_p//K^* \\ (U_{V_p}, V_p) &\rightarrow (U_W \cap U_p, U_p) \times_{U_p//K^*} V_p//K^* \end{aligned}$$

to be the isomorphisms of the toroidal embeddings and hence  $\eta_p$  and  $i_p$  coincide with the obvious étale toroidal morphisms

$$\begin{array}{ccc}
 (T_{X_p}, X_p) & \xleftarrow{\eta_p} & (U_{V_p}, V_p) \\
 \| & & \| \\
 (T_{X_p}, X_p) \times_{X_p//K^*} X_p//K^* & \longleftarrow & (T_{X_p}, X_p) \times_{X_p//K^*} V_p//K^* \\
 (U_{V_p}, V_p) & \xrightarrow{i_p} & (U_W \cap U_p, U_p) \\
 \| & & \| \\
 (U_W \cap U_p, U_p) \times_{U_p//K^*} V_p//K^* & \longrightarrow & (U_W \cap U_p, U_p) \times_{U_p//K^*} U_p//K^*.
 \end{array}$$

We call such a set of strongly étale (toroidal) morphisms **Luna's toroidal chart at p** for the toroidal  $K^*$ -action.

**Proposition 1-3-4.** Any  $K^*$ -action on a nonsingular variety  $W$  is locally toric, i.e., for every closed point  $p \in W$  we can find Luna's locally toric chart at  $p$ .

**Remark 1-3-5.**

We give two alternative proofs for Proposition 1-3-4:

A) One which uses the canonical resolution of singularities and Luna's Fundamental Lemma. In this proof, we show that Luna's locally toric chart

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p$$

can be taken so that  $i_p$  is an isomorphism, i.e.,  $V_p = U_p$  is a  $K^*$ -invariant Zariski open neighborhood of  $p$  and that  $X_p$  is nonsingular. Moreover, we show that in case  $p \in W$  is not a fixed point Luna's locally toric chart can be chosen so that none of  $U_p = V_p$  or  $X_p$  have any fixed points.

B) One which uses Luna's Étale Slice Theorem. In this proof, we show that Luna's locally toric chart can be taken so that  $i_p$  is surjective in general and  $X_p$  is nonsingular and that  $i_p$  is an isomorphism if  $p \in W$  is a fixed point of the  $K^*$ -action. Moreover, we show that in case  $p \in W$  is not a fixed point Luna's locally toric chart can be chosen so that none of  $U_p$ ,  $V_p$  or  $X_p$  has any fixed points.

While Luna's Fundamental Lemma is characteristic free, the canonical resolution of singularities (for the moment) and Luna's Étale Slice Theorem are only valid in characteristic 0.

*Proof.*

A) By Sumihiro's equivariant completion theorem (cf. Sumihiro [1][2]), we can embed  $W$   $K^*$ -equivariantly into a complete variety  $\overline{W}$  with a  $K^*$ -action. By taking the canonical resolution of singularities, which is a sequence of blowups with centers outside of  $W$  (cf. the condition  $(\spadesuit^{res} - 0)$  of the canonical resolution in Remark 4-1-1), we may assume  $\overline{W}$  is nonsingular as well. Note that the  $K^*$ -action lifts to the canonical resolution by the condition  $(\spadesuit^{res} - 1)$ .

Let  $p \in W$  be a closed point. Since  $\overline{W}$  is complete, there exists a fixed point  $q \in \overline{O(p)}$  in the closure of the orbit of  $p$ . By another theorem of Sumihiro, there exists a  $K^*$ -invariant affine open neighborhood  $q \in V_q = \text{Spec } A(V_q)$ . Since the

maximal ideal  $m_q \subset A(V_q)$  associated to the fixed point  $q$  is  $K^*$ -invariant, it splits into a direct sum of eigenspaces

$$m_q = \bigoplus_{\alpha \in \mathbb{Z}} m_{q,\alpha}$$

where  $m_{q,\alpha}$  consists of all the eigenfunctions (in  $m_q$ ) of  $K^*$ -character  $\alpha$ . Therefore, we can choose eigenfunctions

$$f_1, \dots, f_n \in m_q \quad (n = \dim W)$$

with  $K^*$ -characters  $\alpha_1, \dots, \alpha_n$ , i.e.,

$$t^*(f_j) = t^{\alpha_j} \cdot f_j \text{ for } t \in K^*,$$

such that they form a basis of  $m_q/m_q^2$  and hence that they generate the maximal ideal over  $\mathcal{O}_{W,q}$ . Consider the morphism

$$\eta : V_q = \text{Spec } A(V_q) \rightarrow X = \mathbb{A}^n = \text{Spec } [z_1, \dots, z_n]$$

defined by

$$\eta^*(z_1) = f_1, \dots, \eta^*(z_n) = f_n.$$

Letting  $t \in K^*$  act on  $\mathbb{A}^n$  by  $t^*(z_j) = t^{\alpha_j} \cdot z_j$ , i.e., letting  $K^*$  act as the one parameter subgroup  $a = (\alpha_1, \dots, \alpha_n) \in N$  on  $\mathbb{A}^n$  regarded as a standard toric variety, and shrinking  $V_q$  if necessary, we see that  $\eta$  is a  $K^*$ -equivariant étale morphism.

**Lemma 1-3-6 (Luna's Fundamental Lemma).** *Let  $G$  be a reductive group acting on affine varieties  $V$  and  $X$ . Let  $\eta : V \rightarrow X$  be a  $G$ -equivariant morphism. Let  $T$  be a closed orbit of  $G$  in  $V$  such that*

- (i)  $\eta$  is étale at some point of  $T$ ,
- (ii)  $\eta(T)$  is closed in  $X$ ,
- (iii)  $\eta$  is injective on  $T$ , and
- (iv)  $V$  is normal along  $T$ .

*Then there are  $G$ -stable open subsets  $V' \subset V$  and  $X' \subset X$ , with  $T \subset V'$ , such that  $\eta|_{V'} : V' \rightarrow X'$  is a strongly étale  $G$ -equivariant morphism of  $V'$  onto  $X'$ .*

First we apply Luna's Fundamental Lemma to  $\eta : V \rightarrow X$  with  $T = \{q\}$  and  $G = K^*$  to observe that for the choice of  $K^*$ -invariant open subsets  $V' \subset V$  and  $X' \subset X$  as above

$$V' \rightarrow X' \times_{X'//K^*} V' // K^*$$

is an isomorphism.

Since  $O(p) \subset V'$ , this implies that  $\eta$  is injective on  $O(p)$ . The morphism  $\eta$  is étale at all points of  $O(p)$ . Let  $J = \{j; f_j(p) \neq 0\} \subset \{1, \dots, n\}$ . By replacing  $V$  and  $X$  with

$$\begin{aligned} & \{v \in V; \prod_{j \in J} f_j(v) \neq 0\} \\ & \{x \in X; \prod_{j \in J} z_j(x) \neq 0\}, \end{aligned}$$

respectively, we may assume  $\eta(O(p))$  is closed in  $X$ , which is still an affine nonsingular toric variety. By assumption  $V \subset W$  is nonsingular. With these conditions satisfied, we apply Luna's Fundamental Lemma secondly to  $\eta : V \rightarrow X$  with  $T = O(p)$  and  $G = K^*$  to have

$$\eta|_{V'} : V' \rightarrow X' (\subset X)$$

being strongly étale and hence

$$V' \rightarrow X' \times_{X'/\!/K^*} V' \!/K^*$$

being an isomorphism.

Case:  $p \in W$  is not a fixed point.

In this case, by the first application of Luna's Fundamental Lemma we conclude that there exists  $j \in J$  with  $f_j(p) \neq 0$  and  $\alpha_j \neq 0$ . By the choice of  $X$ , we see that  $X$  has no fixed points. Thus we see that  $X' \!/K^* \subset X \!/K^*$  is an open subset.

Therefore, we conclude

- (i)  $\eta_p = \eta|_{V'} : V_p = V' \rightarrow X_p = X$  is étale, and
- (ii)  $V_p = V' \rightarrow X' \times_{X'/\!/K^*} V' \!/K^* = X_p \times_{X_p/\!/K^*} V_p \!/K^*$  is an isomorphism.

Thus

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p$$

is Luna's locally toric chart at  $p$ .

Case:  $p$  is a fixed point.

In this case,  $\eta(p)$  is also a fixed point. Let  $F_X$  be the set of fixed points in  $X$  with respect to the  $K^*$ -action. Let

$$\pi_X : X \rightarrow X \!/K^*$$

be the quotient map. Since  $F_X - X'$  is a  $K^*$ -invariant closed subset of  $X$ , we conclude that  $\pi_X(F_X - X')$  is a closed subset of  $X \!/K^*$  and hence that  $\pi_X^{-1}(\pi_X(F_X - X'))$  is a closed subset of  $X$ . Therefore, we conclude that the subsets defined below

$$\begin{aligned} C &:= F_X \cap X' \\ D &:= \pi_X^{-1}(\pi_X(F_X - X')) \cap X' \end{aligned}$$

are disjoint  $K^*$ -invariant closed subsets of  $X'$ . Therefore, by Corollary 1.2 in Mumford-Fogarty-Kirwan [1], there exists a  $K^*$ -invariant function  $f \in A(X')^{K^*}$  such that

$$f \equiv 1 \text{ on } C \quad \& \quad f \equiv 0 \text{ on } D.$$

Then by construction, setting

$$X'_f = \{x \in X'; f(x) \neq 0\},$$

we see that  $\eta(p) \in X'_f$  and that  $X'_f \!/K^* \subset X \!/K^*$  is an open subset. Set  $V'_f = \eta^{-1}(X'_f)$ . Then since

$$V' \rightarrow X' \times_{X'/\!/K^*} V' \!/K^*$$

is an isomorphism, so is

$$V'_f \rightarrow X'_f \times_{X'_f // K^*} V'_f // K^*.$$

Therefore, we conclude

- (i)  $\eta_p = \eta|_{V_p} : V_p = V'_f \rightarrow X_p = X$  is étale, and
- (ii)  $V_p \rightarrow X'_f \times_{X'_f // K^*} V'_f // K^* = X_p \times_{X_p // K^*} V_p // K^*$  is an isomorphism.

Thus

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p$$

is Luna's locally toric chart at  $p$ .

This completes the proof A) for Proposition 1-3-4 using the canonical resolution of singularities and Luna's Fundamental Lemma.

B) We recall Luna's Étale Slice Theorem.

**Theorem 1-3-7 (Luna's Étale Slice Theorem).** *Let  $G$  be a reductive group. Let  $U$  be an affine variety with a  $G$ -action and let  $T$  be a closed orbit of  $G$  in  $U$  along which  $U$  is normal. If  $p \in T$ , then there exists a locally closed  $G_p$ -stable ( $G_p$  is the stabilizer of  $p$  in  $G$ ) affine subvariety  $Z$  of  $U$  with  $p \in Z$  such that  $U_p = G \cdot Z$  is affine open in  $U$  and the  $G$ -equivariant morphism*

$$\iota : G \times_{G_p} Z \rightarrow U_p$$

*is strongly étale. (Note that  $G \times_{G_p} Z$  is by definition the quotient of  $G \times Z$  by the action of  $G_p$  given by*

$$g \cdot (t, z) = (tg^{-1}, g \cdot w) \text{ for } g \in G_p \text{ and } (t, w) \in G \times Z.)$$

*Moreover, if  $U$  is nonsingular, then  $Z$  can be taken to be also nonsingular and we have the following commutative diagram of fiber squares*

$$\begin{array}{ccccc} G \times_{G_p} N_p & \xleftarrow{\text{étale}} & G \times_{G_p} Z & \xrightarrow{\text{étale surjective}} & U_p \\ \downarrow & \square & \downarrow & & \downarrow \\ N_p // G_p & \xleftarrow{\text{étale}} & Z // G_p & \xrightarrow{\text{étale surjective}} & U_p // G, \end{array}$$

where  $N_p$  is the normal vector space to  $Z$  at  $p$ .

Let  $p \in W$  be a closed point. By a theorem of Sumihiro, there exists a  $K^*$ -invariant affine open neighborhood  $U$  of  $p$  such that  $U$  is also contained in  $W - \{\overline{O(p)} - O(p)\}$ . Now  $T = O(p)$  is a closed orbit of  $G = K^*$  in  $U$ , which is nonsingular by assumption. Then applying Luna's Étale Slice Theorem, we obtain strongly étale morphisms

$$\begin{aligned} i_p : V_p &= G \times_{G_p} Z \rightarrow U_p = G \cdot Z \quad (\text{surjective}) \\ \eta_p : V_p &= G \times_{G_p} Z \rightarrow X_p = G \times_{G_p} N_p, \end{aligned}$$

where  $X_p = G \times_{G_p} N_p = K^* \times_{(K^*)_p} N_p$  is easily seen to be a nonsingular affine toric variety with  $K^*$  acting as a one-parameter subgroup, as it acts on the first factor of  $K^* \times_{(K^*)_p} N_p$  by the usual multiplication.

Thus

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p$$

is Luna's locally toric chart at  $p$ .

Moreover, if  $p \in W$  is a fixed point, then  $i_p$  as above is an isomorphism, i.e.,  $V_p = U_p$  is a  $K^*$ -invariant affine neighborhood of  $p$ . If  $p \in W$  is not a fixed point, then by taking  $U$  to be contained in  $W - F_W$  where  $F_W$  is the fixed point set in  $W$ , we see that none of  $X_p, V_p$  or  $U_p$  has any fixed points by construction.

This completes the proofs A) and B) of Proposition 1-3-4.

#### §1-4. Elimination of points of indeterminacy

For the weak factorization problem, we start with a general birational map  $\phi : X_1 \dashrightarrow X_2$  between complete nonsingular varieties. In this section, we show, by the method of elimination of points of indeterminacy, that we only have to deal with the case where  $\phi$  is a projective birational morphism.

**Lemma 1-4-1.** *There is a commutative diagram*

$$\begin{array}{ccc} X'_1 & \xrightarrow{\phi'} & X'_2 \\ g_1 \downarrow & & \downarrow g_2 \\ X_1 & \dashrightarrow^\phi & X_2 \end{array}$$

such that  $g_1$  and  $g_2$  are sequences of blowups with smooth centers disjoint from  $U$  and that  $\phi'$  is a projective birational morphism.

*Proof.*

By Hironaka's theorem on elimination of points of indeterminacy, there is a sequence of blowups  $g_2 : X'_2 \rightarrow X_2$  with smooth centers disjoint from  $U$  such that the birational map  $h := \phi^{-1} \circ g_2 : X'_2 \rightarrow X_1$  is a morphism. By the same theorem, there is a sequence of blowups  $g_1 : X'_1 \rightarrow X_1$  with smooth centers disjoint from  $U$  such that the birational map  $\phi' : X'_1 \rightarrow X_2$  is a morphism.

Since the composite  $h \circ \phi'$  is projective, we conclude that  $\phi'$  is also projective.

We only have to replace  $\phi : X_1 \dashrightarrow X_2$  by  $\phi' : X'_1 \rightarrow X'_2$  as above to see the reduction Step 1 in the strategy for the proof described in Chapter 0. Introduction.

**Remark 1-4-2 (Elimination of Points of Indeterminacy).**

We should make a couple of remarks about Hironaka's method of elimination of points of indeterminacy for a birational map  $\phi : X_1 \dashrightarrow X_2$ .

First we may assume that  $\phi^{-1}$  is a morphism. Otherwise, replace  $W_2$  by the graph of  $\phi$ . (If one wants to stay in the nonsingular category, take a resolution of singularities of the graph.)

Secondly, if  $\phi^{-1}$  is a projective morphism then we can find an ideal  $I$ , trivial over  $U$ , such that  $\phi^{-1}$  is the blowup of  $X_1$  along  $I$ . (See, for example, the proof of Theorem 2-2-2.) In general, including the case where  $\phi^{-1}$  is not a projective morphism, Hironaka's version of Chow's Lemma (cf. Hironaka [3]) asserts that there exists an ideal  $I$  on  $X_1$  such that the blowup of  $X_1$  along  $I$  factors through

$X_2$ . (Remark that although it is not explicitly stated in Hironaka [3], the ideal  $I$  can be taken so that it is trivial over  $U$ . Remark also that although Hironaka [3] works specifically over the field  $\mathbb{C}$  of complex numbers, the canonicity of his method and Lefschetz principle imply that it is actually valid over any field of characteristic zero (cf. Chapter 4 and Chapter 5). See also Section 5 in Abramovich-Karu-Matsuki-Włodarczyk [1].) We only have to take  $g_1 : X'_1 \rightarrow X_1$  to be a principalization of the ideal  $I$  by a sequence of blowups with smooth centers (lying over the support of  $\mathcal{O}_{X_2}/I$ ) in order to obtain a morphism  $h = \phi \circ g_1 : X'_1 \rightarrow X_2$ .

## CHAPTER 2. BIRATIONAL COBORDISM

In this chapter, we study the theory of birational cobordisms by Włodarczyk [2] (See also Morelli [1].), which is the main tool for our construction of the factorization of a given birational map  $\phi : X_1 \dashrightarrow X_2$  between nonsingular complete varieties. The power of the theory lies in the fact that it allows us to analyze the structure of birational transformations much like the usual Morse theory allows us to study the structure of homotopy transformations via the usual cobordisms. The theory may be considered an algebraic version of the Morse theory in terms of the action of the multiplicative group  $K^*$  through this analogy, which also justifies the definition we give below. (See the introduction in Chapter 0.)

It may be worthwhile to note that the connection between the Morse theory and the action of  $K^*$  on an algebraic variety has been known and may even be considered classical, as was observed by many authors (e.g. Brion-Processi [1] Dolgachev-Hu [1] Frankel [1] Kirwan [1] Mumford-Fogarty-Kirwan [1] Thaddeus [1] [2]). When  $K = \mathbb{C}$  is the field of complex numbers, it is also studied in the realm of symplectic geometry. Actually we take advantage of the interpretation of the birational cobordisms in terms of Geometric Invariant Theory to show that, if both  $X_1$  and  $X_2$  are projective, then we can choose our factorization so that all the intermediate varieties are also projective. The subject of the change of the G.I.T. quotients associated to the change of linearizations has been found to have a close connection with the factorization problem of certain birational maps, as studied by Thaddeus and others.

The novelty of the ideas by Włodarczyk, therefore, is in connecting the long-standing factorization problem of GENERAL birational maps to these classical ideas by constructing birational cobordisms and hence providing a view point from the Morse theory to approach the problem.

We cannot probably emphasize too much the importance of the original work of Morelli [1], who introduced the notion of combinatorial cobordisms to solve the factorization problem of toric birational maps. It is the geometric interpretation of Morelli's combinatorial objects that ultimately led Włodarczyk to the theory of birational cobordism for general birational maps.

### §2-1. Definition of a birational cobordism and the toric main example

**Definition 2-1-1 (Birational Cobordism).** Let  $\phi : X_1 \dashrightarrow X_2$  be a proper birational map between normal varieties defined over  $K$ , isomorphic over a common open subset  $X_1 \supset U \subset X_2$ . A normal variety  $B$  is called a **birational cobordism** for  $\phi$  and denoted by  $B_\phi(X_1, X_2)$  if it satisfies the following conditions:

- (i) the multiplicative group  $K^*$  acts on  $B = B_\phi(X_1, X_2)$ , (We denote the action of  $t \in K^*$  on  $x \in B$  by  $t(x)$  or  $t \cdot x$ .)
- (ii) the sets

$$B_+ := \{x \in B; \lim_{t \rightarrow \infty} t(x) \text{ does NOT exist in } B\}$$

$$B_- := \{x \in B; \lim_{t \rightarrow 0} t(x) \text{ does NOT exist in } B\}$$

are nonempty Zariski open subsets of  $B$ , and

(iii) there are isomorphisms

$$\begin{aligned} B_+/K^* &\xrightarrow{\sim} X_2 \\ B_-/K^* &\xrightarrow{\sim} X_1 \end{aligned}$$

so that the birational map induced by the inclusions  $B_- \supset B_- \cap B_+ \subset B_+$  and the isomorphisms above

$$X_1 = B_-/K^* \supset (B_- \cap B_+)/K^* \subset B_+/K^* = X_2$$

coincides with  $\phi$ .

We say that **B respects the open subset  $\mathbf{U}$**  if  $U$  is contained in  $B_- \cap B_+/K^*$ .

We discuss the following fundamental example of a birational cobordism in the toric setting, as was first observed by Morelli [1].

### Main Toric Example 2-1-2.

Let  $B = \mathbb{A}^n = \text{Spec } K[z_1, \dots, z_n]$  with a  $K^*$ -action given by

$$t(z_1, \dots, z_j, \dots, z_n) = (t^{\alpha_1} z_1, \dots, t^{\alpha_j} z_j, \dots, t^{\alpha_n} z_n).$$

We regard  $\mathbb{A}^n = X(N, \sigma)$  as a toric variety defined by a lattice  $N \cong \mathbb{Z}^n$  and a regular cone  $\sigma \subset N_{\mathbb{R}}$  generated by the standard  $\mathbb{Z}$ -basis of  $N$

$$\sigma = \langle v_1, \dots, v_j, \dots, v_n \rangle.$$

The dual cone  $\sigma^\vee$  is generated by the dual  $\mathbb{Z}$ -basis

$$\sigma^\vee = \langle v_1^*, \dots, v_j^*, \dots, v_n^* \rangle$$

and we identify

$$z_j = z^{v_j^*}.$$

The  $K^*$ -action then corresponds to the one parameter subgroup

$$a = (\alpha_1, \dots, \alpha_j, \dots, \alpha_n) \in N.$$

We have then the obvious description of the sets  $B_+$  and  $B_-$

$$\begin{aligned} B_+ &= \{(z_1, \dots, z_n); z_j \neq 0 \text{ for some } j \text{ with } \alpha_j = (v_j^*, a) > 0\} \\ B_- &= \{(z_1, \dots, z_n); z_j \neq 0 \text{ for some } j \text{ with } \alpha_j = (v_j^*, a) < 0\}. \end{aligned}$$

We define the upper boundary and lower boundary of  $\sigma$  (with respect  $a \in N$ ) to be

$$\begin{aligned} \partial_+\sigma &= \{x \in \sigma; x + \epsilon \cdot (-a) \notin \sigma \text{ for } \epsilon > 0\} \\ \partial_-\sigma &= \{x \in \sigma; x + \epsilon \cdot a \notin \sigma \text{ for } \epsilon > 0\}. \end{aligned}$$

Then we obtain the description of  $B_+$ ,  $B_-$  and  $B_+ \cap B_-$  as the toric varieties corresponding to the fans  $\partial_+\sigma$ ,  $\partial_-\sigma$  and  $\partial_+\sigma \cap \partial_-\sigma$ , i.e.,

$$\begin{aligned} B_+ &= X(N, \partial_+\sigma) \\ B_- &= X(N, \partial_-\sigma). \\ B_+ \cap B_- &= X(N, \partial_+\sigma \cap \partial_-\sigma) \end{aligned}$$

Accordingly, if we denote by  $\pi : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R} \cdot a$  the projection onto  $N_{\mathbb{R}}/\mathbb{R} \cdot a$  with the lattice  $\pi(N)$ , then we have the description of the quotients as the toric varieties

$$\begin{aligned} B_+/K^* &= X(\pi(N), \pi(\partial_+\sigma)) \\ B_-/K^* &= X(\pi(N), \pi(\partial_-\sigma)) \\ (B_+ \cap B_-)/K^* &= X(\pi(N), \pi(\partial_+\sigma \cap \partial_-\sigma)) \\ B//K^* &= X(\pi(N), \pi(\sigma)) \end{aligned}$$

where  $\pi(\partial_+\sigma), \pi(\partial_-\sigma), \pi(\partial_+\sigma \cap \partial_-\sigma)$  and  $\pi(\sigma)$  are fans in  $N_{\mathbb{R}}/\mathbb{R} \cdot a$  defined by

$$\begin{aligned} \pi(\partial_+\sigma) &= \{\pi(\tau); \tau \in \partial_+\sigma\} \\ \pi(\partial_-\sigma) &= \{\pi(\tau); \tau \in \partial_-\sigma\} \\ \pi(\partial_+\sigma \cap \partial_-\sigma) &= \{\pi(\tau); \tau \in \partial_+\sigma \cap \partial_-\sigma\} = \pi(\partial_+\sigma) \cap \pi(\partial_-\sigma) \\ \pi(\sigma) &= \{\pi(\sigma)\} \text{ and its proper faces in } N_{\mathbb{R}}/\mathbb{R} \cdot a, \end{aligned}$$

respectively. Since  $\pi(\partial_+\sigma)$  and  $\pi(\partial_-\sigma)$  are subdivisions of  $\pi(\sigma)$ , we obtain a diagram of toric birational maps

$$\begin{array}{ccc} B_-/K^* & \xrightarrow{-\phi} & B_+/K^* \\ \searrow & & \swarrow \\ \parallel & & \parallel \\ X(\pi(N), \pi(\partial_-\sigma)) & \dashrightarrow & X(\pi(N), \pi(\partial_+\sigma)) \\ \searrow & & \swarrow \\ \parallel & & \parallel \\ X(\pi(N), \pi(\sigma)) & & \end{array}$$

where  $\phi$  coincides with the birational map induced by

$$\begin{array}{ccccc} B_-/K^* & \supset & (B_- \cap B_+)/K^* & \subset & B_+/K^* \\ \parallel & & \parallel & & \parallel \\ X(\pi(N), \pi(\partial_-\sigma)) & \supset & X(\pi(N), \pi(\partial_-\sigma \cap \partial_+\sigma)) & \subset & X(\pi(N), \pi(\partial_+\sigma)). \end{array}$$

Therefore, we conclude that  $B$  is a birational cobordism for  $\phi$ .

More generally, one can prove (See Morelli [1] Abramovich-Matsuki-Rashid [1].) that if  $\Sigma$  is a subdivision of a convex polyhedral cone in  $N_{\mathbb{R}}$  with the lower boundary  $\partial_-\Sigma$  and upper boundary  $\partial_+\Sigma$  with respect to a one-parameter subgroup  $a \in N$ , then the toric variety  $X(N, \Sigma)$  corresponding to  $\Sigma$  with the  $K^*$ -action given by the one-parameter subgroup  $a \in N$ , is a birational cobordism between the two toric varieties  $X(\pi(N), \pi(\partial_-\Sigma))$  and  $X(\pi(N), \pi(\partial_+\Sigma))$ .

Next in the pursuit of analogy to the usual Morse theory where the critical points are “lined up nicely” according to the levels given by the Morse function, we would like to have the fixed points of a birational cobordism “ordered nicely” so that we can study the birational transformations as we go through the fixed points “from the bottom to the top”, though we may not see a Morse function

explicitly. This requirement naturally leads us to the notion of “**collapsibility**” introduced by Morelli [1]. In the toric setting that Morelli [1] studied, the fixed points of the cobordism, associated to a fan  $\Sigma$ , correspond to the cones  $\sigma$  of the maximal dimension (bubbles) in  $\Sigma$ . Factorization then corresponds to going from  $\partial_-\Sigma$  to  $\partial_+\Sigma$ , by replacing  $\partial_-\sigma$  with  $\partial_+\sigma$  one at a time (collapsing of the bubble  $\sigma$ ). (See the main toric example 2-1-2 and Morelli [1] Abramovich-Matsuki-Rashid [1].) Thus the question of whether or not we can order the fixed points nicely corresponds to that of whether or not we can collapse these bubbles in a nicely ordered manner, thus giving rise to the name “collapsibility”.

First we introduce the notations for some specific subsets in  $B$  associated to the fixed point set.

**Notation 2-1-3.** Let  $B = B_\phi(X_1, X_2)$  be a birational cobordism, and let  $F \subset F_B = B^{K^*}$  be a subset of the fixed point set with respect to the  $K^*$ -action. We define

$$\begin{aligned} F^+ &:= \{x \in B; \lim_{t \rightarrow 0} t(x) \in F\} \\ F^- &:= \{x \in B; \lim_{t \rightarrow \infty} t(x) \in F\} \\ F^\pm &:= F^+ \cup F^- \\ F^* &:= F^\pm - F. \end{aligned}$$

**Definition 2-1-4.** Let  $B = B_\phi(X_1, X_2)$  be a birational cobordism. We define a relation  $\prec$  among the connected components of the fixed point set  $B^{K^*}$  as follows: let  $F_1, F_2 \subset B^{K^*}$  be two (not necessarily distinct) connected components, and set  $F_1 \prec F_2$  if there exists a point  $p \in B$  with  $p \notin B^{K^*}$  such that

$$\lim_{t \rightarrow 0} t(p) \in F_1 \text{ and } \lim_{t \rightarrow \infty} t(p) \in F_2.$$

That is to say, in terms of Notation 2-1-3, we have the relation  $F_1 \prec F_2$  if and only if  $(F_1^+ - F_1) \cap (F_2^- - F_2) \neq \emptyset$ .

**Definition 2-1-5 (Collapsibility).** We say that a birational cobordism  $B = B_\phi(X_1, X_2)$  is collapsible if the relation  $\prec$  is a strict preorder, namely, there is no directed cycle of the connected components of the fixed point set

$$F_1 \prec F_2 \prec \cdots \prec F_m \prec F_1.$$

Note that collapsibility excludes the possibility of a self-loop  $F_1 \prec F_1$  for any connected component  $F_1 \subset B^{K^*}$ .

In Chapter 3, we will consider the toric ideals on a birational cobordism  $B$  where, however, the existence of a point  $p \in B$  with  $p \notin B^{K^*}$  such that both limits  $\lim_{t \rightarrow 0} t(p)$  and  $\lim_{t \rightarrow \infty} t(p)$  exist within  $B$  would cause a problem in order to have such ideals well-defined. The notion of a quasi-elementary birational cobordism below is introduced exactly to avoid this problem.

**Definition 2-1-6 (Quasi-Elementary Birational Cobordism).** A birational cobordism  $B$  is said to be quasi-elementary if there does not exist any point  $p \in B$  with  $p \notin B^{K^*}$  such that both limits  $\lim_{t \rightarrow 0} t(p)$  and  $\lim_{t \rightarrow \infty} t(p)$  exist within  $B$ . In terms of the relation  $\prec$ , this is equivalent to saying that for any two (not necessarily distinct) connected components  $F_1, F_2 \subset B^{K^*}$  neither  $F_1 \prec F_2$  nor  $F_2 \prec F_1$  holds. That is to say, in terms of Notation 2-1-3,  $B$  is quasi-elementary if and only if  $(F^+ - F) \cap (F^- - F) = \emptyset$  where  $F = B^{K^*}$  is the entire fixed point set in  $B$ .

**Definition 2-1-7 (Elementary Birational Cobordism).** A quasi-elementary birational cobordism is said to be elementary if the fixed point set  $B^{K^*}$  is connected.

We will observe that the birational transformation represented by an elementary birational cobordism corresponds to a (weighted) blowup of a connected center, which is étale locally equivalent to a toric birational transformation. (See §2-4 for the details.) One might want to call such a birational transformation “elementary”, as Włodarczyk [2] does, and hence name the corresponding birational cobordism also “elementary”.

## §2-2. Construction of a (collapsible) birational cobordism

After defining a birational cobordism, the natural and important issue is its existence, which Włodarczyk [2] shows for any proper birational map between (normal) varieties. Here we present a simple construction of a collapsible birational cobordism for a projective birational morphism  $\phi : X_1 \rightarrow X_2$  between complete nonsingular varieties, which suffices for our purposes after reducing the factorization problem of an arbitrary birational map to that of a projective morphism via Hironaka’s theorem elimination of indeterminacy. See §2-4 for the details of the reduction step.

**Theorem 2-2-2 (Construction of Birational Cobordism).** Let  $\phi : X_1 \rightarrow X_2$  be a projective (in the sense of Grothendieck and not as defined in Hartshorne [1]) birational morphism between complete nonsingular varieties (defined over  $K$ ), which is an isomorphism over a common open subset  $U$ . Then there exists a complete nonsingular variety  $\overline{B}$  with an effective  $K^*$ -action, (Note that a  $K^*$ -action is called **effective** if the action is not induced from that of the nontrivial quotient of  $K^*$ , that is to say, if  $\cap_{p \in \overline{B}} \text{Stab}(p) = \{1\}$ .) satisfying the following properties:

(i) there exist closed embeddings

$$\begin{aligned} i_1 : X_1 &\hookrightarrow \overline{B} \\ i_2 : X_2 &\hookrightarrow \overline{B} \end{aligned}$$

with disjoint images in  $\overline{B}^{K^*}$ ,

(ii) there is a coherent sheaf  $\mathcal{E}$  on  $X_2$ , with a  $K^*$ -action (which is compatible with the trivial action of  $K^*$  on  $X_2$ ), and a  $K^*$ -equivariant closed embedding

$$\overline{B} \hookrightarrow \mathbb{P}(\mathcal{E}) := \text{Proj}_{X_2} \oplus_{m \geq 0} \text{Sym}^m \mathcal{E},$$

(iii) the open subvariety  $B := \overline{B} - (i_1(X_1) \cup i_2(X_2))$  is a collapsible birational cobordism for  $\phi$ .

We call such a variety  $\overline{B}$  a **compactified birational cobordism** projective over  $X_2$ .

**Remark 2-2-3.**

Suppose that a projective birational morphism between complete nonsingular varieties  $\phi : X_1 \rightarrow X_2$  is a sequence of blowups with smooth centers. Then we can construct a compatified birational cobordism  $\overline{B}$  projective over  $X_2$  in the following simple manner: We start with the product  $W_0 = X_2 \times \mathbb{P}^1$  where  $K^*$  acts on the second factor as the multiplication on  $K^* = \mathbb{P}^1 - \{0, \infty\}$ . We take the sequence of blowups  $W \rightarrow X_2 \times \mathbb{P}^1$  with the centers identified with those for  $\phi$  but considered as lying in (the strict transforms of) the 0-section  $X_2 \times \{0\}$  instead of lying in (the subsequent blowups of)  $X_2$ . Now  $\overline{B} = W$  is a complete nonsingular variety projective over  $X_2$ , with an effective  $K^*$ -action, satisfying the properties:

- (i) we have two closed embeddings

$$\begin{aligned} i_1 : X_1 &\xrightarrow{\sim} (\text{the strict transform of } X_2 \times \{0\}) \subset \overline{B} \\ i_2 : X_2 &\xrightarrow{\sim} X_2 \times \{\infty\} \subset \overline{B} \end{aligned}$$

with disjoint images  $i_1(X_1), i_2(X_2) \subset \overline{B}^{K^*}$

(ii) the centers of blowups lie in the fixed point set of the  $K^*$ -action and hence the morphism  $\overline{B} \rightarrow X_2 \times \mathbb{P}^1 \rightarrow X_2$  is obviously  $K^*$ -equivariant, which easily implies the existence of such a coherent sheaf  $\mathcal{E}$  as above,

(iii)  $B = \overline{B} - (i_1(X_1) \cup i_2(X_2))$  is a collapsible birational cobordism for  $\phi$ . Remark that the factorization induced by  $B$ , as will be discussed in §2-4, coincides with the given sequence of blowups with smooth centers.

The above birational cobordism can be considered as the standard birational cobordism associated to a sequence of blowups with smooth centers factoring a projective birational morphism  $\phi$  (if such a sequence exists at all for  $\phi$ ). The construction below for the proof of Theorem 2-2-2 may be considered as modeled on that of the standard birational cobordism discussed above.

*Proof of Theorem 2-2-2.*

We start with the product  $W_0 = X_2 \times \mathbb{P}^1$  where  $K^*$  acts on the second factor as the multiplication on  $K^* = \mathbb{P}^1 - \{0, \infty\}$ .

Since  $\phi$  is a projective birational morphism which is an isomorphism over  $U$ , there exists an ideal sheaf  $J \subset \mathcal{O}_{X_2}$  such that  $\phi : X_1 \rightarrow X_2$  is the blowup morphism of  $X_2$  along  $J$  and that the support of  $\mathcal{O}_{X_2}/J$  is disjoint from  $U$ . (This can be verified as follows: Take a divisor  $D$  on  $X_1$  which is  $\phi$ -ample. Then we see  $D - \phi^*\phi_*D$  is also  $\phi$ -ample and that it is of the form  $\Sigma(-a_i)E_i$  where  $f_*D$  is the cycle-theoretic image of  $D$  under  $\phi$  as a divisor and where the  $E_i$  are exceptional divisors with  $a_i > 0$ . We only have to take  $J = \phi_*\mathcal{O}_{X_1}(l \cdot \Sigma(-a_i)E_i)$  for some sufficiently large  $l \in \mathbb{N}$ . Note that in our case the existence of such a  $\phi$ -ample divisor  $\Sigma(-a_i)E_i$  follows from the construction. (cf. §5-1.) Remark also that in our construction of the factorization the projective morphism  $\phi$  is obtained as  $\phi' : X'_1 \rightarrow X'_2$  starting from the original birational map  $\phi : X_1 \dashrightarrow X_2$  via the elimination of the points of indeterminacy. Thus in the notation of Lemma 1-4-1 we have a  $g_1$ -ample divisor of the form  $\Sigma(-b_j)F_j$ , where  $F_j$  are  $g_1$ -exceptional divisors whose centers on  $X_1$  lie outside of  $U$ . The divisor  $\Sigma(-b_j)F_j$  is hence  $\phi$ -ample. We can take  $J = \phi_*\mathcal{O}_{X_1}(l \cdot \Sigma(-b_j)F_j)$  for some sufficiently large  $l \in \mathbb{N}$ .) Let  $I_0$  be the ideal of the origin  $0 \in \mathbb{P}^1$ . We set

$$I = (p_1^{-1}J + p_2^{-1}I_0) \cdot \mathcal{O}_{W_0}$$

where

$$\begin{aligned} p_1 : W_0 &= X_2 \times \mathbb{P}^1 \rightarrow X_1 \\ p_2 : W_0 &= X_2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \end{aligned}$$

are the projections.

Let  $W$  be the blowup of  $W_0$  along  $I$ .

We claim that  $X_1$  and  $X_2$  are embedded in the nonsingular locus of  $W$ , as the strict transform of  $X_2 \times \{0\}$  and  $X_2 \times \{\infty\}$ , respectively. For  $i_2 : X_2 \xrightarrow{\sim} X_2 \times \{\infty\} \subset W$  this is clear, as the centers of blowup for  $X_2 \times \mathbb{P}^1$  only lie over  $X_2 \times \{0\}$ . For  $i_2 : X_1 \xrightarrow{\sim} (\text{the strict transform of } X_2 \times \{0\})$ , in order to prove that  $X_1$  is in the nonsingular locus of  $W$ , it suffices to show that  $X_1$  is a Cartier divisor as  $X_1$  itself is nonsingular. We look at local coordinates. Let  $A(V)$  be the affine coordinate ring for an affine open subset  $V \subset X_2$  and let  $y_1, \dots, y_m$  be a set of generators for  $J$  on  $V$ . Let  $K[x]$  be the affine coordinate ring for  $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$ . Then the affine open subset  $V \times \mathbb{A}^1 \subset X_2 \times \mathbb{P}^1$  has the affine coordinate ring  $A(V) \otimes_K K[x]$ , where the ideal  $I$  is generated by  $y_1, \dots, y_m, x$ . The charts of the blowup containing the strict transform of  $V \times \{0\} = \{x = 0\}$  are of the form

$$\text{Spec } A[\frac{y_1}{y_i}, \dots, \frac{y_m}{y_i}, \frac{x}{y_i}] = \text{Spec } A[\frac{y_1}{y_i}, \dots, \frac{y_m}{y_i}] \times \text{Spec } K[\frac{x}{y_i}],$$

where  $K^*$  acts on the second factor by the multiplication on  $x$ . The strict transform of  $\{x = 0\}$  is defined by  $\frac{x}{y_i}$  and hence is Cartier.

Let  $\overline{B} \rightarrow W$  be the canonical resolution of singularities. Since the centers of blowups of the canonical resolution of singularities are taken over the singular locus (cf. the condition ( $\spadesuit^{res} - 0$ )) and since the (effective)  $K^*$ -action lifts to  $\overline{B}$  (cf. the condition ( $\spadesuit^{res} - 1$ )), the above analysis of  $W$  immediately implies the properties (i) and (ii), except for the collapsibility of  $B$ .

Remark that the morphism  $\tau : \overline{B} \rightarrow X_2$  is  $K^*$ -equivariant as well as projective. Therefore, there exists a relatively ample line bundle  $\mathcal{L}$  on  $\overline{B}$  over  $X_2$ , equipped with a  $K^*$ -action (compatible with the  $K^*$ -action on  $\overline{B}$ ). (Remark that in our case the existence of such a relatively ample line bundle follows directly from the construction (cf. §5-1).) In order to see the property (iii), we only have to set

$$\mathcal{E} = \tau_*(\mathcal{L}^{\otimes l}) \text{ for sufficiently large } l \in \mathbb{N}.$$

Finally we show that the birational cobordism  $B$  constructed as above is collapsible.

Let  $\mathcal{C}$  be the set of the connected components of  $B^{K^*}$ . It suffices to show that there exists a strictly increasing function  $\chi : \mathcal{C} \rightarrow \mathbb{Z}$ , i.e., a function such that

$$F_1 \prec F_2 \longrightarrow \chi(F_1) < \chi(F_2).$$

Since  $K^*$  acts trivially on  $X_2$  and since  $K^*$  is reductive, there exists a direct sum decomposition

$$\mathcal{E} = \bigoplus_{b \in \mathbb{Z}} \mathcal{E}_b$$

where  $\mathcal{E}_b$  is the subsheaf on which the  $K^*$ -action is given by the character  $t \mapsto t^b$ . Denote by

$$b_0 < b_1 < \cdots < b_{h-1} < b_h$$

the characters which appear in this representation. Note that there are disjoint embeddings  $\mathbb{P}(\mathcal{E}_b) \subset \mathbb{P}(\mathcal{E})$ .

Let  $p \in B$  be a point lying in the fiber  $\mathbb{P}(E_q)$  over  $q \in X_2$ , where  $E_q = \mathcal{E} \otimes \mathcal{O}_{X_2, q}/m_{X_2, q}$ . We choose a basis of  $E_q$ , which are then considered to be homogeneous coordinates of  $\mathbb{P}(E_q)$

$$(x_{b_0,1}, \dots, x_{b_0,d_0}, \dots, x_{b_h,1}, \dots, x_{b_h,d_h})$$

where  $x_{b_j,k} \in E_{b_j} = \mathcal{E}_{b_j} \otimes \mathcal{O}_{X_2, q}/m_{X_2, q}$ .

It is straightforward to observe that  $p \in B$  is a fixed point if and only if

$$\exists b_{j_p} \text{ such that } x_{b_j,k}(p) = 0 \text{ whenever } b_j \neq b_{j_p}.$$

Therefore, if  $F \subset B^{K^*}$  is a connected component, then  $F$  is contained in  $\mathbb{P}(\mathcal{E}_{b_j})$  for some  $b_j$ . We define

$$\chi(F) := b_j.$$

Suppose  $F_1 \prec F_2$ , i.e., there exists a point  $p \in B$  but  $p \in B^{K^*}$  with

$$\lim_{t \rightarrow 0} t(p) \in F_1 \text{ and } \lim_{t \rightarrow \infty} t(p) \in F_2.$$

It is easy to see that

$$\begin{aligned} \lim_{t \rightarrow 0} t(p) &\in \mathbb{P}(E_{b_{min}}) \\ \lim_{t \rightarrow \infty} t(p) &\in \mathbb{P}(E_{b_{max}}), \end{aligned}$$

where

$$\begin{aligned} b_{min} &= \min\{b_j; x_{b_j,k}(p) \neq 0 \text{ for some } k\} \\ b_{max} &= \max\{b_j; x_{b_j,k}(p) \neq 0 \text{ for some } k\}. \end{aligned}$$

Observe also that

$$b_{min} < b_{max}$$

since  $p \in B$  is not a fixed point. Thus we conclude that

$$\chi(F_1) = b_{min} < b_{max} = \chi(F_2),$$

showing  $\chi$  is a strictly increasing function.

Therefore, the birational cobordism  $B$  is collapsible.

This completes the proof of Theorem 2-2-2.

### §2-3. Interpretation by Geometric Invariant Theory

In this section, we apply Geometric Invariant Theory to the compactified birational cobordism  $\overline{B}$  projective over  $X_2$  and see that the birational cobordism  $B$

decomposes into the quasi-elementary pieces  $B_{a_i}$  (See below for the precise definition), which are the semi-stable loci of the different linearizations of the  $K^*$ -action on  $\overline{B}$ . Thus we can analyze the birational transformations of  $B_{a_i}/K^*$  as the change of the G.I.T. quotients when we vary the linearizations, a subject which is extensively studied by Thaddeus [1][2] and others.

We continue using the notation of Theorem 2-2-2.

Let  $\mathcal{E}$  be a coherent sheaf on  $X_2$ , with a  $K^*$ -action (which is compatible with the trivial action of  $K^*$  on  $X_2$ ), and a  $K^*$ -equivariant closed embedding

$$\overline{B} \hookrightarrow \mathbb{P}(\mathcal{E}) := \text{Proj}_{X_2} \oplus_{m \geq 0} \text{Sym}^m \mathcal{E}$$

and let

$$\mathcal{E} = \bigoplus_{b \in \mathbb{Z}} \mathcal{E}_b$$

be its decomposition into the eigen-subsheaves according to their characters. Let

$$a_1 < a_2 < \cdots < a_{s-2} < a_{s-1}$$

be the values of  $\chi : \mathcal{C} \rightarrow \mathbb{Z}$ , i.e.,  $\{a_i\}$  is the subset of  $\{b_j\}$ , which are the values associated to the connected components of the fixed point set  $B^{K^*}$ . By using the Veronese embedding  $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}(\text{Sym}^2 \mathcal{E})$  and replacing  $\mathcal{E}$  with  $\text{Sym}^2 \mathcal{E}$ , we may assume that all the  $a_i$  are even and that in particular

$$a_{i-1} < a_i - 1 < a_i < a_i + 1 < a_{i+1} \text{ for } i = 2, \dots, s-2.$$

(This is a technical condition which avoids the rational “twists” and allows us only to deal with the integral ones.)

Denote by

$$\rho_0(t)$$

the original  $K^*$ -action on  $\mathcal{E}$ . For any  $r \in \mathbb{Z}$  we consider the “twisted” action of  $K^*$  on  $\mathcal{E}$ , denoted by

$$\rho_r(t) = t^{-r} \cdot \rho_0(t),$$

so that in the decomposition  $\mathcal{E} = \bigoplus_{b \in \mathbb{Z}} \mathcal{E}_b$  the  $\rho_r(t)$  acts on  $\mathcal{E}_b$  by the multiplication  $t^{b-r}$ . Note that the “twists” do not change the induced action of  $K^*$  on  $\mathbb{P}(\mathcal{E})$  but only change the linearizations on  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ .

We can now apply Geometric Invariant Theory in its relative form (cf. Hu [1] Pandripande [1]) to the situation.

We denote by

$$(\mathbb{P}(\mathcal{E}), \rho_r)^{ss}$$

the semi-stable locus of  $\mathbb{P}(\mathcal{E})$  with respect to the linearization of  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  induced by the twisted action  $\rho_r$  of  $K^*$  on  $\mathcal{E}$ . Recall that a point  $p \in \mathbb{P}(\mathcal{E})$  is semi-stable, i.e.,  $p \in (\mathbb{P}(\mathcal{E}), \rho_r)^{ss}$  if there exists a local section  $s \in \text{Sym}^n \mathcal{E}$  for some positive  $n \in \mathbb{N}$ , invariant under the twisted action  $\rho_r$  of  $K^*$  on  $\text{Sym}^n \mathcal{E}$ , such that  $s(p) \neq 0$ . We denote by

$$(\overline{B}, \rho_r)^{ss}$$

the semi-stable locus of  $\overline{B}$  with respect to the linearization of  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\overline{B}}$  coming from that of  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  induced by the twisted action  $\rho_r$  of  $K^*$ . Then we have

$$(\overline{B}, \rho_r)^{ss} = \overline{B} \cap (\mathbb{P}(\mathcal{E}), \rho_r)^{ss}$$

by Theorem 1.19 in Kirwan-Fogarty-Mumford [1].

Now Thaddeus and others tell us that we should observe the change of the semi-stable loci when the twists pass through critical values and that we should also observe the birational transformations of the G.I.T. quotients.

In our situtaion, these critical values turn out to be just

$$a_1 < a_2 < \dots < a_{s-2} < a_{s-1}$$

and the change of the semi-stable loci provides the decomposition of the birational cobordism  $B$  into the quasi-elementary pieces.

**Proposition 2-3-1.** *Let the situation be as above. Then*

(o) *the critical values for the change of the semi-stable loci are*

$$a_1 < a_2 < \dots < a_{s-2} < a_{s-1},$$

(i) *at each critical value  $a_i$  the semi-stable locus  $(\overline{B}, \rho_{a_i})^{ss} := B_{a_i}$  is a quasi-elementary birational cobordism and we have set-theoretic descriptions*

$$\begin{aligned} B_{a_i} &= (\overline{B}, \rho_{a_i})^{ss} \\ &= B - \{(\cup_{F \in \mathcal{C}, \chi(F) < a_i} F^-) \cup (\cup_{F \in \mathcal{C}, \chi(F) > a_i} F^+)\} \\ (B_{a_i})_+ &= (\overline{B}, \rho_{a_i+1})^{ss} \\ (B_{a_i})_- &= (\overline{B}, \rho_{a_i-1})^{ss}, \end{aligned}$$

(ii) *we have the commutative diagarm of (proper) birational morphsims and maps between varieties PROJECTIVE over  $X_2$*

$$\begin{array}{ccc} (B_{a_i})_- / K^* & \xrightarrow{\varphi_i} & (B_{a_i})_+ / K^* \\ \parallel & & \parallel \\ (\overline{B}, \rho_{a_i-1})^{ss} / K^* & \dashrightarrow & (\overline{B}, \rho_{a_i+1})^{ss} / K^* \\ \searrow & \parallel & \swarrow \\ & (\overline{B}, \rho_{a_i})^{ss} // K^* & \end{array}$$

(iii) *we also have*

$$\begin{aligned} (B_{a_1})_- &= B_- \text{ and } (B_{a_1})_- / K^* = B_- / K^* = X_1 \\ (B_{a_{s-1}})_+ &= B_+ \text{ and } (B_{a_{s-1}})_+ / K^* = B_+ / K^* = X_2. \end{aligned}$$

*Proof.*

The proof is an easy consequence of the (relative) G.I.T. and the definition of the semi-stable locus as above and left to the reader as an exercise.

#### §2-4. Factorization into locally toric birational maps

As a summary of the results discussed in the previous sections §2-1, §2-2 and §2-3 and the use of elimination of points of indeterminacy by Hironaka, we obtain the following factorization theorem of a birational map into locally toric transformations by Włodarczyk [2].

**Theorem 2-4-1 (Factorization into Locally Toric Birational Maps).** *Let  $\phi : X_1 \rightarrow X_2$  be a birational map between complete nonsingular varieties over  $K$ , which is an isomorphism over a common open subset  $X_1 \supset U \subset X_2$ . Then there exists a sequence of birational maps*

$$X_1 = W_1 \dashrightarrow^{\varphi_1} W_2 \dashrightarrow^{\varphi_2} \cdots \dashrightarrow^{i-1} W_i \dashrightarrow^{\varphi_i} W_i \dashrightarrow^{i+1} \cdots \dashrightarrow^{s-1} W_s = X_2$$

such that

- (i)  $\phi = \varphi_{s-1} \circ \varphi_{s-2} \circ \cdots \circ \varphi_2 \circ \varphi_1$ ,
- (ii)  $\varphi_i$  is  $V$ -locally toric and an isomorphism over  $U$ , and that
- (iii) there exists an index  $i_o$  with the property that for all  $i \leq i_o$  the map  $W_i \dashrightarrow X_1$  is a projective birational morphism and for all  $i \geq i_o$  the map  $W_i \dashrightarrow X_2$  is a projective birational morphism, and hence if  $X_1$  and  $X_2$  are projective then all the  $W_i$  are projective.

*Proof.*

By the reduction step as discussed in §1-4, we may assume that  $\phi : X_1 \rightarrow X_2$  is a projective birational morphism. Thus we are in a situation to apply Theorem 2-2-2 to obtain a compactified birational cobordism  $\overline{B} \subset \mathbb{P}(\mathcal{E})$  projective over  $X_2$ , where  $\mathcal{E}$  is a coherent sheaf on  $X_2$  with a  $K^*$ -action. The associated birational cobordism  $B = \overline{B} - (i_1(X_1) \cup i_2(X_2))$  is collapsible and the connected components  $F$  of its fixed points are ordered by the values of a strictly increasing function  $\chi : \mathcal{C} \rightarrow \mathbb{Z}$

$$a_1 < a_2 < \cdots < a_{s-2} < a_{s-1}.$$

By looking at the semi-stable loci with respect to the linearizations induced by the “twisted” actions of  $K^*$  on  $\mathcal{E}$  and at the corresponding G.I.T. quotients, we obtain a factorization

$$\begin{array}{ccc} W_i = (B_{a_i})_- / K^* & \dashrightarrow^{\varphi_i} & W_{i+1} = (B_{a_i})_+ / K^* \\ \searrow & & \swarrow \\ \parallel & B_{a_i} // K^* & \parallel \\ (\overline{B}, \rho_{a_i-1})^{ss} / K^* & \dashrightarrow & (\overline{B}, \rho_{a_i+1})^{ss} / K^* \\ \searrow & \parallel & \swarrow \\ & (\overline{B}, \rho_{a_i})^{ss} // K^* & \end{array}$$

starting with  $X_1 = B_- / K^* = (B_{a_1})_- / K^* = W_1$  and ending with  $W_l = (B_{a_l})_+ / K^* = B_+ / K^* = X_2$ , as given in Proposition 2-3-1. By construction, each  $W_i$  is projective

over  $X_2$  and  $\varphi_i$  is an isomorphism over  $U$ . The existence of the index  $i_o$  as claimed in (iii) is also clear, once one takes the reduction in Step 0 into consideration.

Thus it remains to show that each  $\varphi_i$  is a V-locally toric birational map.

**Lemma 2-4-4.** *Let  $B_{a_i}$  be a quasi-elementary cobordism. Then for every closed point  $p \in B_{a_i} - F_{B_{a_i}}^*$ , where  $F_{B_{a_i}} = B_{a_i}^{K^*}$  and  $F_{B_{a_i}}^*$  is the set as defined in Notation 2-1-3, there exists Luna's locally toric chart*

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p \subset B_{a_i}$$

such that the following condition  $(\star)$  is satisfied:

$(\star)$  If a  $K^*$ -orbit  $O(q)$  lies in  $U_p$ , then its closure  $\overline{O(q)}$  in  $B_{a_i}$  also lies in  $U_p$ .

*Proof.*

We take Luna's locally toric chart

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p \subset B_{a_i}$$

as constructed in Proposition 1-3-4, which may not satisfy the condition  $(\star)$ . Let  $\pi : B_{a_i} \rightarrow B_{a_i} // K^*$  be the quotient map.

Case:  $p \in B_{a_i} - F_{B_{a_i}}^*$  is not a fixed point.

In this case,  $p \in B_{a_i} - \pi^{-1}(\pi(F_{B_{a_i}}))$ . Thus by shrinking  $U_p$  if necessary, we may assume  $U_p \subset B_{a_i} - \pi^{-1}(\pi(F_{B_{a_i}}))$ , in which case the condition  $(\star)$  is automatically satisfied.

Case:  $p \in B_{a_i} - F_{B_{a_i}}^*$  is a fixed point.

In this case, set

$$\begin{aligned} C &:= F_{B_{a_i}} \cap U_p \\ D &:= \pi^{-1}(\pi(F_{B_{a_i}} - U_p)) \cap U_p = (F_{B_{a_i}} - U_p)^\pm \cap U_p. \end{aligned}$$

Since  $C$  and  $D$  are disjoint  $K^*$ -invariant closed subsets of an affine  $K^*$ -invariant variety  $U_p$ , Corollary 1.2 in Kirwan-Fogarty-Mumford [1] implies that there exists a  $K^*$ -invariant function  $f \in A(U_p)^{K^*}$  such that

$$f \equiv 1 \text{ on } C \quad \& \quad f \equiv 0 \text{ on } D.$$

We only have to replace the original  $U_p$  and  $V_p$  with

$$\begin{aligned} (U_p)_f &= \{q \in U_p; f(q) \neq 0\} \text{ and} \\ (V_p)_f &= \{v \in V_p; i_p^* f(v) \neq 0\}, \end{aligned}$$

respectively, in order for Luna's locally toric chart to satisfy the condition  $(\star)$ .

We resume the proof for  $\varphi_i$  being V-locally toric.

Remark that  $\{U_p\}_{p \in B_{a_i} - F_{B_{a_i}}^*}$  where we have Luna's locally toric chart

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p \subset B_{a_i}$$

satisfying the condition  $(\star)$  is an open covering of  $B_{a_i}$ . By the condition  $(\star)$  the morphism  $V_p//K^* \rightarrow B_{a_i}//K^*$  is étale, as it is a composite of an étale morphism  $V_p//K^* \rightarrow U_p//K^*$  with an inclusion  $U_p//K^* \subset B_{a_i}//K^*$ . We also have the commutative diagram

$$\begin{array}{ccccccc}
 (B_{a_i})_-/K^* & \longrightarrow & B_{a_i}//K^* & \longleftarrow & (B_{a_i})_+/K^* \\
 \uparrow & \square & \uparrow & \square & \uparrow \\
 (U_p)_-/K^* \times_{U_p//K^*} V_p//K^* \rightarrow & V_p//K^* & \leftarrow (U_p)_+/K^* \times_{U_p//K^*} V_p//K^* \\
 \parallel & & \parallel & & \parallel \\
 (V_p)_-/K^* & \longrightarrow & V_p//K^* & \longleftarrow & (V_p)_+/K^* \\
 \parallel & & \parallel & & \parallel \\
 (X_p)_-/K^* \times_{X_p//K^*} V_p//K^* \rightarrow & V_p//K^* & \leftarrow (X_p)_+/K^* \times_{X_p//K^*} V_p//K^* \\
 \downarrow & \square & \downarrow & \square & \downarrow \\
 (X_p)_-/K^* & \longrightarrow & X_p//K^* & \longleftarrow & (X_p)_+/K^*.
 \end{array}$$

Therefore, according to Definition 1-2-10 (i), the proper birational map  $\varphi_i$  is V-locally toric.

This completes the proof of Theorem 2-4-1.

### CHAPTER 3. TRIFICATION

In the previous chapter, we observed that the  $K^*$ -action on the entire cobordism  $B$  and hence on the quasi-elementary cobordisms  $B_{a_i}$  is locally toric and that this gives rise to the  $V$ -locally toric birational map

$$\begin{array}{ccc}
 W_i & \xrightarrow{\varphi_i} & W_{i+1} \\
 \| & & \| \\
 (B_{a_i})_-/K^* & & (B_{a_i})_+/K^* \\
 \searrow & & \swarrow \\
 & (B_{a_i})/K^* &
 \end{array}$$

The reason why  $\varphi_i : W_i = (B_{a_i})_-/K^* \dashrightarrow W_{i+1} = (B_{a_i})_+/K^*$  is merely locally toric and not toroidal is that the choices of the local coordinate divisors (for Luna's locally toric charts for  $B_{a_i}$ ) are not canonical and hence that they do NOT patch together to provide a global boundary divisor  $B_{a_i} - U_{B_{a_i}}$  necessary for a desired toroidal structure  $(U_{B_{a_i}}, B_{a_i})$ .

In order to create such a global boundary divisor, we blow up  $B_{a_i}$  along an ideal  $I$ , called the “**torific**” ideal on  $B_{a_i}$ , canonically defined in terms of the locally toric  $K^*$ -action on  $B_{a_i}$ . We show that the divisor  $D^{tor}$  defined by the principal ideal  $\mu^{-1}(I) \cdot B_{a_i}^{tor}$ , where

$$\mu : B_{a_i}^{tor} \rightarrow B_{a_i}$$

is the normalization of the blowup of  $B_{a_i}$  along  $I$ , provides the toroidal structure  $(U_{B_{a_i}^{tor}} := B_{a_i}^{tor} - D^{tor}, B_{a_i}^{tor})$  with respect to which the induced  $K^*$ -action on  $B_{a_i}^{tor}$  is toroidal. We call such a procedure “**torification**”.

The basic idea that if one blows up an ideal then the divisors defined by the principalized ideal provide the resulting variety with some useful extra structure, can be traced back at least to Hironaka. The torific ideal we construct is closely related to the one used by Abramovich-DeJong [1] for their proof of Weak Resolution of Singularities.

After torification,  $B_{a_i}^{tor}$  turns out to be a quasi-elementary cobordism and hence we obtain a  $V$ -toroidal birational map

$$\begin{array}{ccc}
 (U_{W_{i-}^{tor}}, W_{i-}^{tor}) & \xrightarrow{\varphi_i^{tor}} & (U_{W_{i+}^{tor}}, W_{i+}^{tor}) \\
 \| & & \| \\
 (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_-/K^* & & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_+/K^* \\
 \searrow & & \swarrow \\
 & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^* &
 \end{array}$$

Remark in general that  $W_i, W_{i\pm}^{tor}$  are all singular and that the induced morphisms  $f_{i\pm} : W_{i\pm}^{tor} \rightarrow W_i$  are not sequences of blowups with smooth centers on nonsingular

varieties. Thus it seems that through the birational cobordism we decomposed  $\phi$  into a sequence of locally toric birational maps, which were made then toroidal via torification, but at the fatal cost of introducing singularities. In Chapter 4, we remedy this situation by applying canonical resolution of singularities to  $W_i, W_{i\pm}^{tor}$  (and canonical principalization of some ideals on them) so that we can apply the strong factorization theorem of toroidal birational maps between nonsingular toroidal embeddings by Morelli [1] (cf. Włodarczyk [1] Abramovich-Matsuki-Rashid [1]) to finish the proof of the Weak Factorization Theorem.

The ideal situation would have been that we could define the torific ideal on the entire cobordism  $B$ , the blowup of the torific ideal would then torify the  $K^*$ -action and hence we could reduce the factorization of  $\phi$  to that of ONE toroidal birational map. But at the moment, for a small but essential reason to be discussed below, we can only define the torific ideal on each quasi-elementary cobordism  $B_{a_i}$  separately but not on the entire cobordism. Thus we are forced to break up  $\phi$  into SEVERAL toroidal birational maps and cannot make the toroidal structures compatible with each other. This is the main reason why we can only prove the WEAK Factorization Theorem but not the STRONG one (for the moment).

In this chapter,  $B = B_{a_i}$  always refers to a nonsingular quasi-elementary cobordism by abuse of notation with  $K^*$  acting effectively. We also omit the subscript  $a_i$  from the corresponding notations, e.g.,  $F = F_{a_i}, F^* = F_{a_i}^*$  etc. This convention is restricted to this chapter only.

### §3-1. Construction of the torific ideal

**Definition 3-1-1.** Let  $p \in B$  be a closed point with  $G_p \subset K^*$  being the stabilizer of  $p$ . Fix an integer  $\alpha \in \mathbb{Z}$ . Then we define

$$J_{\alpha,p} \subset \mathcal{O}_{B,p}$$

to be the ideal generated by all the semi-invariant functions  $f \in \mathcal{O}_{B,p}$  of  $G_p$ -character  $\alpha$ , that is,

$$t^*(f) = t^\alpha \cdot f \text{ for } t \in G_p \subset K^*.$$

**Theorem 3-1-2.** There exists a unique coherent  $K^*$ -equivariant ideal sheaf  $I_\alpha \subset \mathcal{O}_B$  such that

$$(I_\alpha)_p = J_{\alpha,p} \quad \forall p \in B - F^*.$$

(We note that a  $K^*$ -equivariant sheaf is nothing but a  $K^*$ -sheaf in the sense of Mumford [1], where he considers the notion of a  $G$ -sheaf when a group  $G$  acts on a variety. In our case,  $\mathcal{O}_B$  has the natural structure of a  $K^*$ -sheaf induced by the  $K^*$ -action on  $B$  and we require that the ideal  $I_\alpha$  has the  $K^*$ -subsheaf structure.)

**Definition 3-1-3.** The coherent sheaf  $I_\alpha$  as above is called the  $\alpha$ -torific ideal sheaf with respect to the  $K^*$ -action on  $B$ .

**Remark 3-1-4.**

- (i) It is absolutely necessary to exclude the points in  $F^*$  from the ones on which we require the condition  $(I_\alpha)_p = J_{\alpha,p}$ . The collection of ideals  $J_{\alpha,p}$  for  $p \in B$  does

not define a coherent sheaf in general. As an example, let  $B = \mathbb{A}^2$  and let  $K^*$  act on it by

$$t \cdot (x, y) = (tx, t^{-1}y) \text{ for } t \in K^*.$$

We fix  $\alpha = 1$ . Then at  $p = (0, 0)$  the stabilizer is  $G_p = K^*$  and  $J_{1,p} = (x)$ . On the other hand, at any other point  $q \in B - \{p\}$ , the stabilizer is trivial  $G_q = \{1\} \subset K^*$  and hence  $J_{1,q} = \mathcal{O}_{B,q}$ . While  $J_{1,q} = \mathcal{O}_{B,q}$  for  $q = (x, y)$  with  $x \neq 0$  and  $J_{1,p} = (x)$  patch together,  $J_{1,q} = \mathcal{O}_{B,q}$  for  $q = (x, y)$  with  $x = 0$  and  $J_{1,p} = (x)$  do NOT.

(ii) For a point  $q \in F^*$ , the stalk  $(I_\alpha)_q$  is determined by the  $K^*$ -equivariance of  $I_\alpha$  and by the fact that the closure  $\overline{\mathcal{O}(q)}$  of the orbit of  $q$  has a unique fixed point  $p \in F \subset B - F^*$ , since  $B$  is quasi-elementary. (See the proof for the uniqueness of the  $\alpha$ -toric ideal.) On the other hand, if  $B$  is not quasi-elementary, the closure  $\overline{\mathcal{O}(q)}$  may have two distinct fixed points  $p_1$  and  $p_2$  and the possible stalks at  $q$  induced from those at  $p_1$  and  $p_2$ , respectively, by the  $K^*$ -equivariance may not coincide. This is why we have trouble defining the toric ideal if  $B$  is not quasi-elementary.

*Proof of Theorem 3-1-2.*

**(Uniqueness)** Suppose we have two coherent ideal sheaves  $I_\alpha$  and  $I'_\alpha$  satisfying the condition of Theorem 3-1-2. Then

$$(I_\alpha)_p = (I'_\alpha)_p = J_{\alpha,p} \text{ for } p \in B - F^*.$$

We have to check

$$(I_\alpha)_q = (I'_\alpha)_q \text{ for } q \in F^*.$$

Since  $B$  is quasi-elementary, the closure  $\overline{\mathcal{O}(q)}$  contains only one fixed point  $r \in \overline{\mathcal{O}(q)}$ . Since  $r \in F \subset B - F^*$ , we have

$$(I_\alpha)_r = (I'_\alpha)_r = J_{\alpha,r}.$$

Since both  $I_\alpha$  and  $I'_\alpha$  are coherent, there exists an open neighborhood  $r \in U$  such that

$$I_\alpha|_U = I'_\alpha|_U.$$

Take a point  $q_o = t_o \cdot q \in \mathcal{O}(q) \cap U$ . Then by the  $K^*$ -equivariance we have

$$(I_\alpha)_q = t_o^* \{(I_\alpha)_{q_o}\} = t_o^* \{(I'_\alpha)_{q_o}\} = (I'_\alpha)_q,$$

proving the uniqueness.

**(Existence)**

Step 1. First observe that it is enough to prove the following.

**Claim 3-1-5.** *For each point  $p \in B - F^*$ , there exist a  $K^*$ -invariant open neighborhood  $p \in U_p$  satisfying the condition (\*):*

(\*). *If a  $K^*$ -orbit  $\mathcal{O}(q)$  lies in  $U_p$ , then its closure  $\overline{\mathcal{O}(q)}$  also lies in  $U_p$ .*

and the  $\alpha$ -toric ideal over  $U_p$ , i.e., a  $K^*$ -equivariant coherent ideal sheaf  $I_{\alpha,U_p}$  over  $U_p$  such that

$$(I_{\alpha,U_p})_q = J_{\alpha,q} \quad \forall q \in U_p \cap (B - F^*).$$

In fact, the collection  $\{U_p\}_{p \in B - F^*}$  is an open covering of  $B$  and the collection of ideals  $\{I_{\alpha, U_p}\}_{p \in B - F^*}$  patches together by the uniqueness argument, using the condition (\*), to provide the global  $\alpha$ -toric ideal over  $B$  as required.

Step 2. We take Luna's locally toric chart for  $p \in B - F^*$

$$X \xleftarrow{\eta} V \xrightarrow{i} Y = U_p \subset B$$

where  $\eta$  and  $i$  are strongly étale with  $i$  being surjective,  $X_p$  is a nonsingular affine toric variety and where  $Y = U_p$  satisfies the condition (\*).

**Claim 3-1-6.** *In order to show the existence of the  $\alpha$ -toric ideal  $I_{\alpha, U_p}$ , it suffices to show the existence of the  $\alpha$ -toric ideal  $I_{\alpha, X}$  over the toric variety  $X$ .*

We spend the rest of Step 2 for the verification of this claim.

**Lemma 3-1-7.** *Let  $p \in B$  and  $J_{\alpha, p} \subset \mathcal{O}_{B, p}$  be the ideal generated by all the semi-invariant functions in  $\mathcal{O}_{B, p}$  of  $G_p$ -character  $\alpha$  as in Definition 3-1-1. We lift the  $G_p$ -action on  $\mathcal{O}_{B, p}$  to the one on the completion  $\widehat{\mathcal{O}}_{B, p}$ . Then the completion  $\widehat{J}_{\alpha, p}$  of the ideal  $J_{\alpha, p}$*

$$\widehat{J}_{\alpha, p} = J_{\alpha, p} \otimes \widehat{\mathcal{O}_{B, p}} \subset \mathcal{O}_{B, p} \otimes \widehat{\mathcal{O}_{B, p}} = \widehat{\mathcal{O}_{B, p}}$$

*is characterized as the ideal generated by all the semi-invariant functions in  $\widehat{\mathcal{O}_{B, p}}$  of  $G_p$ -character  $\alpha$  with respect to the induced  $G_p$ -action on  $\widehat{\mathcal{O}_{B, p}}$ .*

*Proof.*

The affine coordinate ring for  $Y = \text{Spec } A$  splits into the direct sum

$$A = \bigoplus_{\beta \in \mathbb{Z}} A_\beta$$

where the  $A_\beta$  consist of all the semi-invariant functions of  $K^*$ -character  $\beta$ . This splitting induces the splitting of the maximal ideal  $m_p$  for the point  $p$

$$m_p = \bigoplus_{\beta' \in \mathbb{Z}/b_p} m_{p, \beta'}$$

where  $m_{p, \beta'}$  are the elements of  $m_p$  with  $G_p$ -character  $\beta'$ . (The stabilizer  $G_p \subset K^*$  is identified with the group of  $b_p$ -th roots of unity. We set  $b_p = 0$  when  $G_p = K^*$ .) In fact, it is easy to see that any direct summand  $A_\beta$  must be contained in  $m_p$  if  $\beta \pmod{b_p} \neq 0$ . Thus taking  $m_{p, 0} = m_p \cap \bigoplus_{\beta \pmod{b_p} = 0} A_\beta$  and  $m_{p, \beta'} = \bigoplus_{\beta \pmod{b_p} = \beta'} A_\beta$  for  $\beta' \neq 0$  we obtain the above splitting. Therefore, we can choose semi-invariant functions  $z_1, \dots, z_n$  (with respect to the  $G_p$ -action) which form a basis of the vector space  $m_p/m_p^2$  and hence generates  $m_p$  by Nakayama's Lemma. We have a  $G_p$ -equivariant isomorphism

$$\widehat{\mathcal{O}_{B, p}} \cong K[[z_1, \dots, z_n]]$$

where the action of  $G_p$  on the formal power series is induced by the  $G_p$ -characters on  $z_1, \dots, z_n$ .

Since the  $G_p$ -action respects the natural grading on  $K[[z_1, \dots, z_n]]$  given by the degree and since  $K[[z_1, \dots, z_n]]$  is noetherian, the ideal  $K_\alpha$  generated by all

the semi-invariant functions in  $\widehat{\mathcal{O}_{B,p}}$  of  $G_p$ -character  $\alpha$  is generated by a finite number of monomials  $m_1, \dots, m_l$  of  $G_p$ -character  $\alpha$  in  $z_1, \dots, z_n$ . Since obviously  $m_1, \dots, m_l \in J_{\alpha,p}$ , we have

$$K_\alpha = (m_1, \dots, m_l) \subset \widehat{J_{\alpha,p}}.$$

Since  $J_{\alpha,p} \subset K_\alpha$  by definition and hence  $\widehat{J_{\alpha,p}} \subset K_\alpha$ , we conclude

$$K_\alpha = \widehat{J_{\alpha,p}}$$

as required.

We resume the verification of Claim 3-1-6.

For any point  $q \in Y$ , let  $q_V \in i^{-1}(q)$  be a point in the inverse image of  $q$  by  $i$  and  $q_X = \eta(q_V)$ . (We note that  $q_V$  and hence  $q_X$  may not be determined uniquely only by specifying the point  $q$ .) Observe by the condition (\*) that we have

$$q \in B - F^* \iff q_X \in X - F_X^*$$

and in particular

$$p \in B - F^* \implies p_X \in X - F_X^*.$$

Let  $\mathcal{H}_\alpha \subset \mathcal{O}_Y$  be the ideal sheaf over  $Y$  generated by the monomials  $m_1, \dots, m_l$  as taken in the proof of Lemma 3-1-7. Then we have

$$(\eta^* \widehat{I_{\alpha,X}})_{p_V} = \eta^*((\widehat{I_{\alpha,X}})_{p_X}) \xleftarrow{\sim} (\widehat{I_{\alpha,X}})_{p_X}$$

and

$$(i^* \widehat{\mathcal{H}_\alpha})_{p_V} = i^*((\widehat{\mathcal{H}_\alpha})_p) \xleftarrow{\sim} (\widehat{\mathcal{H}_\alpha})_p = \widehat{J_{\alpha,p}}.$$

Now since  $I_{\alpha,X}$  is the  $\alpha$ -toric ideal over  $X$  and since  $p_X \in X - F_X^*$ , Lemma 3-1-7 characterizes both  $(\eta^* \widehat{I_{\alpha,X}})_{p_V}$  and  $(i^* \widehat{\mathcal{H}_\alpha})_{p_V}$  as the ideal generated by all the semi-invariant functions in  $\widehat{\mathcal{O}_{V,p_V}}$  of  $G_{p_V}$ -character  $\alpha$ . (Remark that  $G_{p_V} = G_{p_X} = G_p$  since  $\eta$  and  $i$  are both strongly étale.) Therefore, we have

$$(\eta^* \widehat{I_{\alpha,X}})_{p_V} = (i^* \widehat{\mathcal{H}_\alpha})_{p_V},$$

which implies (See, e.g., Matsumura [1] Theorem 8.15 (3).)

$$(\eta^* I_{\alpha,X})_{p_V} = (i^* \mathcal{H}_\alpha)_{p_V}.$$

Therefore, noting that the equality above holds for all the points  $p_V \in i^{-1}(p)$  and by shrinking  $U_p = Y$  and accordingly  $V$  if necessary, we may assume

$$\eta^* I_{\alpha,X} = i^* \mathcal{H}_\alpha.$$

Now for any point  $q \in Y - F^*$  we have  $q_X \in X - F_X^*$  and

$$i^*(\widehat{J_{\alpha,q}}) = \eta^*((\widehat{I_{\alpha,X}})_{q_X}),$$

both sides being characterized as the ideal generated by all the semi-invariant functions of  $G_{qV}$  ( $= G_q = G_{qX}$ )-character  $\alpha$  in  $\widehat{\mathcal{O}_{V,qV}}$  ( $\cong \widehat{\mathcal{O}_{B,q}} \cong \widehat{\mathcal{O}_{X,qX}}$ ). On the other hand, we have

$$\begin{aligned}\eta^*((\widehat{I_{\alpha,X}})_{qX}) &= (\widehat{\eta^* I_{\alpha,X}})_{qV} \\ &= (\widehat{i^* \mathcal{H}_\alpha})_{qV} \\ &= i^*(\widehat{(\mathcal{H}_\alpha)_q}).\end{aligned}$$

Since  $i$  is étale, we conclude

$$\widehat{J_{\alpha,q}} = \widehat{(\mathcal{H}_\alpha)_q},$$

which implies

$$J_{\alpha,q} = (\mathcal{H}_\alpha)_q.$$

Since  $\eta$  and  $i$  are  $K^*$ -equivariant and since  $I_{\alpha,X}$  is  $K^*$ -equivariant as it is the  $\alpha$ -toric ideal over  $X$ , the equality  $\eta^* I_{\alpha,X} = i^* \mathcal{H}_\alpha$  implies that  $\mathcal{H}_\alpha$  is also  $K^*$ -equivariant.

Therefore, we conclude that we can take  $\mathcal{H}_\alpha$  to be the  $\alpha$ -toric ideal  $I_{\alpha,U_p}$  over  $Y = U_p$ .

Step 3. We show the existence of the  $\alpha$ -toric ideal on an affine nonsingular toric variety  $X$  where  $K^*$ -acts as a one-parameter subgroup.

**Proposition 3-1-8.** *Let  $X = X(N, \sigma)$  be an affine nonsingular toric variety where  $K^*$  acts as a one-parameter subgroup. Then the ideal generated by all the monomials of  $K^*$ -character  $\alpha$  is the  $\alpha$ -toric ideal over  $X$ .*

*Proof.*

We take a  $\mathbb{Z}$ -basis  $\{v_1, \dots, v_n\}$  of the lattice  $N$  so that

$$\begin{aligned}\sigma &= \langle v_1, \dots, v_l \rangle \\ \sigma^\vee &= \langle v_1^*, \dots, v_l^*, \pm v_{l+1}^*, \dots, \pm v_n^* \rangle.\end{aligned}$$

Setting  $z_j = z^{v_j^*}$ , we have

$$X = \text{Spec } K[z_1, \dots, z_l, z_{l+1}^{\pm 1}, \dots, z_n^{\pm 1}].$$

The  $K^*$ -action corresponds to a one-parameter subgroup

$$a = (\alpha_1, \dots, \alpha_n) \in N,$$

i.e., the action of  $t \in K^*$  is given by

$$t \cdot (z_1, \dots, z_n) = (t^{\alpha_1} z_1, \dots, t^{\alpha_n} z_n).$$

Let  $H_{\alpha,X}$  be the ideal generated by all the monomials of  $K^*$ -character  $\alpha$ . Then we see obviously that  $H_{\alpha,X}$  is  $K^*$ -equivariant and that

$$(H_{\alpha,X})_q \subset J_{\alpha,q} \text{ for } q \in X - F_X^*.$$

Thus in order to prove  $H_{\alpha,X}$  is the  $\alpha$ -torific ideal over  $X$  we only have to show the opposite inclusion

$$J_{\alpha,q} \subset (H_{\alpha,X})_q.$$

Remark first that  $J_{\alpha,q}$  is generated by all the monomials in  $z_1, \dots, z_n$  of  $G_q$ -character  $\alpha$ . (Note that  $z_1 - z_1(q), \dots, z_n(q) - z_n(q)$  are generators of the maximal ideal  $m_q$  and are all semi-invariant functions with respect to the action of  $G_q$ . Indeed, if  $z_i(q) \neq 0$ , then  $z_i$  has the trivial  $G_q$ -character. Then using the argument in the proof of Lemma 3-1-7, we see that  $J_{\alpha,q}$  is generated by all the monomials in  $z_1 - z_1(q), \dots, z_n(q) - z_n(q)$  of  $G_q$ -character  $\alpha$  and hence generated by all the monomials in  $z_1, \dots, z_n$  of  $G_q$ -character  $\alpha$ ).

Let  $f \in \mathcal{O}_{X,q}$  be a monomial in  $z_1, \dots, z_n$  of  $G_q$ -character  $\alpha$ . We claim that there exists a monomial  $g$  in  $z_1, \dots, z_n$  invertible at  $q$ , i.e.,  $g \in \mathcal{O}_{X,q}^*$ , such that  $f \cdot g$  is a monomial of  $K^*$ -character  $\alpha$ .

In fact, let  $b_q \in \mathbb{Z}_{\geq 0}$  be the non-negative generator of the subgroup of  $\mathbb{Z}$  generated by the following set

$$A_q := \{\alpha_j; \alpha_j \neq 0 \text{ with } z_j(q) \neq 0\}.$$

The stabilizer  $G_q \subset K^*$  consists of the  $b_q$ -th roots of unity. Let  $\alpha_f$  be the  $K^*$ -character of the monomial  $f$ . Then  $\alpha - \alpha_f$  is a multiple of  $b_q$ , say  $b_f b_q$ . By the description of the set  $A_q$ , we have a monomial  $g$  in  $z_1, \dots, z_n$ , invertible at  $q$ , of  $K^*$ -character  $b_f b_q$ . Then  $f \cdot g$  is a monomial of  $K^*$ -character  $\alpha$  by construction.

Since  $f \cdot g$  is a monomial regular at  $q$ , it is regular in a  $K^*$ -invariant open neighborhood  $U_q$  of  $q$ . Since  $X$  is affine and hence quasi-elementary, we conclude that  $U_q$  must contain  $\pi^{-1}(U)$ , where  $\pi : X \rightarrow X//K^*$  is the quotient morphism and where  $U$  is some open neighborhood of  $\pi(q)$ . Take a global function  $h$  on  $X//K$  (which is identified with a global  $K^*$ -invariant function on  $X$ ) such that  $h \equiv 0$  on  $X//K^* - U$  and  $h(\pi(q)) = 1$  (See Mumford-Fogarty-Kirwan [1] Corollary 1.2.). Then  $f \cdot g \cdot h^l$  (for  $l >> 0$ ) is a global regular function of  $K^*$ -character  $\alpha$ , hence a linear combination of monomials in  $z_1, \dots, z_n$  of  $K^*$ -character  $\alpha$ .

Since  $f$  is an arbitrary monomial of  $G_q$ -character  $\alpha$  and since  $g$  and  $h$  are invertible at  $q$ , we conclude that  $J_{\alpha,q}$  generated by all the monomials of  $G_q$ -character  $\alpha$  is also generated by all the monomials of  $K^*$ -character  $\alpha$ . Therefore, we have

$$J_{\alpha,q} \subset (H_{\alpha,X})_q.$$

This completes the proof for Proposition 3-1-8 and hence the proof for Theorem 3-1-2 via Steps 1, 2 and 3.

### §3-2. The torifying property of the torific ideal

In this section, we discuss the torifying property of the torific ideal (which is actually the product of several  $\alpha$ -torific ideals defined in the previous section, where the characters  $\alpha$  are taken from a certain finite set), i.e., how we can induce the torification of the  $K^*$ -action by blowing up the torific ideal, as announced at the beginning of Chapter 3.

First we explain the **torification principle** for affine nonsingular toric varieties  $X$ , showing the torifying property for them. The explanation of the torification

principle should also provide the motivation for the definition of the torific ideal, including that of the  $\alpha$ -torific ideal of the previous section. Secondly, utilizing on Luna's locally toric charts and based upon the torifying property for toric varieties, we discuss the torifying property of the torific ideal on a quasi-elementary cobordism in general.

**Observation 3-2-1 (Torification Principle for Toric Varieties).**

Let  $X = X(N, \sigma)$  be an affine nonsingular toric variety with  $K^*$  acting as a one-parameter subgroup. As in Proposition 3-1-8, we take a  $\mathbb{Z}$ -basis of  $N$  so that

$$\begin{aligned}\sigma &= \langle v_1, \dots, v_l \rangle \\ \sigma^\vee &= \langle v_1^*, \dots, v_l^*, \pm v_{l+1}^*, \dots, \pm v_n^* \rangle \\ X &= \text{Spec } K[z_1, \dots, z_l, z_{l+1}^\pm, \dots, z_n^\pm] \text{ with } z_j = z^{v_j^*}\end{aligned}$$

and that the  $K^*$ -action corresponds to

$$a = (\alpha_1, \dots, \alpha_n) \in N.$$

We have the coordinate divisors

$$D_{v_1} = \{z_1 = 0\}, \dots, D_{v_l} = \{z_l = 0\}$$

corresponding to the 1-dimensional faces (generated by the  $v_i$ ) of  $\sigma$ . As  $X$  is a toric variety containing the torus  $T_X$ , obviously

$$(T_X = X - (D_{v_1} \cup \dots \cup D_{v_l}), X)$$

has the toric and hence toroidal structure with respect to which the  $K^*$ -action is toric and hence toroidal.

Let  $I$  be an ideal generated by some monomials and let

$$\mu_X : \tilde{X} \rightarrow X$$

be the normalization of the blowup of  $X$  along  $I$ . Since  $I$  is toric having monomial generators, the torus-action lifts to the blowup of  $X$  along  $I$  and to the normalization  $\tilde{X}$ , which is thus again a toric variety  $\tilde{X} = X(N, \tilde{\sigma})$  where  $\tilde{\sigma}$  is a fan obtained by subdividing  $\sigma$ . (We define a toric variety to be a NORMAL variety which contains a torus, as a dense open subset, whose action on itself extends to the entire variety.) The  $K^*$ -action also lifts to  $\tilde{X}$  being a one-parameter subgroup.

Let  $\tilde{D}$  to be the divisor defined by the principal ideal  $\mu_X^{-1}(I) \cdot \mathcal{O}_{\tilde{X}}$ . Then as  $\tilde{X}$  is a toric variety containing the torus  $T_X = T_{\tilde{X}}$ ,

$$(T_{\tilde{X}} = \tilde{X} - \{(D_{v_1} \cup \dots \cup D_{v_l}) \cup \tilde{D}\}, \tilde{X})$$

has the toric and hence toroidal structure with respect to which the  $K^*$ -action is toric and hence toroidal. (We use the same notation  $D_{v_i}$  for the strict transform on  $\tilde{X}$  of the divisor  $D_{v_i}$  on  $X$ , indicating both are the closures of the orbit corresponding to the 1-dimensional ray generated by  $v_i$  in  $\tilde{X}$  and  $X$ , respectively.)

We look for a situation on  $\tilde{X}$  (and the conditions on  $I$  which yields that situation on  $\tilde{X}$ ) where

$$(U_{\tilde{X}} = \tilde{X} - \tilde{D}, \tilde{X})$$

has a toroidal structure with respect to which the  $K^*$ -action is toroidal. That is to say, after removing those coordinate divisors among  $D_{v_1}, \dots, D_{v_l}$  which are NOT contained in  $\tilde{D}$  from the toric boundary divisor  $(D_{v_1} \cup \dots \cup D_{v_l}) \cup \tilde{D}$  we still want to preserve a toroidal stucture on  $(U_{\tilde{X}}, \tilde{X})$  with respect to which the  $K^*$ -action remains toroidal.

Recall that the original coordinate divisors  $D_{v_1}, \dots, D_{v_l}$  correspond to a locally toric chart of the quasi-elementary cobordism  $B$  and are not chosen canonically and that we have to remove those which are NOT contained in the canonically constructed boundary divisor, which corresponds to  $\tilde{D}$  which arises from the torific ideal defined canonically only in terms of the  $K^*$ -action.

**Lemma 3-2-2.** *We consider a fixed coordinate divisor  $D_{v_j}$  where  $j = 1, \dots, l$ . Suppose that we have the following situation  $(\heartsuit_j)$ :*

$$(\heartsuit_j) \left\{ \begin{array}{l} \text{For any affine (open) toric subvariety } Z \subset \tilde{X}, \\ \text{we have } (T_Z, Z) = (T_{\mathbb{A}^1}, \mathbb{A}^1) \times (T_{Z'_j}, Z'_j) \text{ as toric varieties,} \\ \text{provided that } D_{v_j} \not\subset \tilde{D} \text{ and } Z \cap D_{v_j} \neq \emptyset, \\ \text{where } \mathbb{A}^1 \text{ has the standard toric structure with torus } T_{\mathbb{A}^1} = \mathbb{A}^1 - \{0\} \\ \text{and } Z'_j \text{ is an affine toric variety with the torus } T_{Z'_j} \\ \text{and where the } K^* \text{-action on } \mathbb{A}^1 = \text{Spec } K[x_j] \text{ is trivial} \\ \text{and } K^* \text{ acts on } Z'_j \text{ as a one-parameter subgroup in } T_{Z'_j} \\ \text{and we have } Z \cap D_{v_j} = \{x_j = 0\}. \end{array} \right.$$

Then

$$(U_{\tilde{X}, j} = \tilde{X} - \{(D_{v_1} \cup \dots \cup \overset{\vee}{D_{v_j}} \cup \dots \cup D_{v_l}) \cup \tilde{D}\}, \tilde{X})$$

has a toroidal structure with respect to which the  $K^*$ -action is toroidal.

*Proof.*

If  $D_{v_j} \subset \tilde{D}$ , then for any affine (open) toric subvariety  $Z \subset \tilde{X}$  we have

$$(T_Z, Z) = Z \cap (U_{\tilde{X}, j}, \tilde{X}) = Z \cap (U_{\tilde{X}}, \tilde{X})$$

has the toric and hence toroidal structure with respect to which the  $K^*$ -action is toric and hence toroidal. Since these  $Z$ 's cover the entire  $\tilde{X}$ , the claim follows.

So we may assume  $D_{v_j} \not\subset \tilde{D}$ . Let  $Z \subset \tilde{X}$  be an affine (open) toric subvariety.

If  $Z \cap D_{v_j} = \emptyset$ , then

$$(T_Z, Z) = Z \cap (U_{\tilde{X}, j}, \tilde{X}) = Z \cap (U_{\tilde{X}}, \tilde{X})$$

has the toric and hence toroidal structure with respect to which the  $K^*$ -action is toric and hence toroidal.

If  $Z \cap D_{v_j} \neq \emptyset$ , then by the situation  $(\heartsuit_j)$  we have

$$\begin{aligned} Z \cap (U_{\tilde{X},j}, \tilde{X}) &= (\mathbb{A}^1, \mathbb{A}^1) \times (U_{Z'_j}, Z'_j) \\ &= (U_{Z_{\{0\}}}, Z_{\{0\}}) \cup (U_{Z_{\{1\}}}, Z_{\{1\}}) \\ &\text{where} \\ (U_{Z_{\{0\}}}, Z_{\{0\}}) &:= (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (U_{Z'_j}, Z'_j) \text{ and} \\ (U_{Z_{\{1\}}}, Z_{\{1\}}) &:= (\mathbb{A}^1 - \{1\}, \mathbb{A}^1 - \{1\}) \times (U_{Z'_j}, Z'_j). \end{aligned}$$

Since the  $K^*$ -action on  $\mathbb{A}^1 = (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \cup (\mathbb{A}^1 - \{1\}, \mathbb{A}^1 - \{1\})$  is trivial, we have

$$(U_{Z_{\{0\}}}, Z_{\{0\}}), (U_{Z_{\{1\}}}, Z_{\{1\}}) \xrightarrow{K^*\text{-equivariant}} (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (U_{Z'_j}, Z'_j).$$

Since  $Z$ 's with  $Z \cap D_{v_j} = \emptyset$  and  $Z_{\{0\}}, Z_{\{1\}}$ 's with  $Z \cap D_{v_j} \neq \emptyset$  cover the entire  $\tilde{X}$ , the claim follows.

**Remark 3-2-3.** Let  $S = \{j; D_{v_j} \in \tilde{D}\}$  be the set of indices of the primitive vectors generating the 1-dimensional cones in  $\tilde{\sigma}$  which correspond to the divisors in  $\tilde{D}$ . Then  $(\heartsuit_j)$  is equivalent to the following  $(\heartsuit_{j,fan})$  in terms of the geometry of the fan  $\tilde{\sigma}$ :

$$(\heartsuit_{j,fan}) \left\{ \begin{array}{l} \text{For any cone } \sigma_Z \in \tilde{\sigma}, \\ \text{we have } \sigma_Z = \langle v_j \rangle \oplus \tau_{Z'_j} \text{ as cones,} \\ \text{provided that } j \notin S \text{ and } v_j \in \sigma_Z, \\ \text{where the cone } \tau_{Z'_j} = \langle w'_k s \rangle \text{ is generated by} \\ \text{the extremal rays } w_k \text{'s (other than those } v_j \text{'s with } j \notin S \text{) of } \sigma_Z \\ \text{and is contained in a hyperplane } H_j \text{ which also contains } a \in N \\ \text{and where we have the exact sequence} \\ 0 \rightarrow H_j \cap N \rightarrow N \rightarrow \mathbb{Z} \rightarrow 0 \\ \text{with the image of } v_j \in N \text{ to be } \pm 1 \in \mathbb{Z}. \end{array} \right.$$

Now it is immediate to see that the blowup of the  $\alpha_j$ -toric ideal  $I_{\alpha_j, X}$  yields the situation  $(\heartsuit_j)$ . (See the proof of Theorem 3-2-6.) This justifies and actually motivated the definition of the  $\alpha$ -toric ideal in the previous section.

In order to see that we can remove all the coordinate divisors which are not contained in  $\tilde{D}$  simultaneously, not just one  $D_{v_j} \not\subset \tilde{D}$  at a time and

$$(U_{\tilde{X}}, \tilde{X})$$

has a toroidal structure, we need the lemma below.

**Lemma 3-2-4.** Suppose we have the following situation ( $\heartsuit$ ):

( $\heartsuit$ ) {

- For any affine (open) toric subvariety  $Z \subset \tilde{X}$ ,
- we have  $(T_Z, Z) = \prod_j (T_{\mathbb{A}_j^1}, \mathbb{A}_j^1) \times (T_{Z'}, Z')$  as toric varieties,
- with the indices  $j$  varying among those with  $D_{v_j} \not\subset \tilde{D}$  and  $Z \cap D_{v_j} \neq \emptyset$ ,
- where  $\mathbb{A}_j^1$  has the standard toric structure with torus  $T_{\mathbb{A}_j^1} = \mathbb{A}_j^1 - \{0\}$
- and  $Z'$  is an affine toric variety with the torus  $T_{Z'}$
- and where the  $K^*$ -action on  $\mathbb{A}_j^1 = \text{Spec } K[x_j]$  is trivial
- and  $K^*$  acts on  $Z'$  as a one-parameter subgroup in  $T_{Z'}$
- and we have  $Z \cap D_{v_j} = \{x_j = 0\}$ .

Then

$$(U_{\tilde{X}} = \tilde{X} - \tilde{D}, \tilde{X})$$

has a toroidal structure with respect to which the  $K^*$ -action is toroidal.

The proof is identical to that of Lemma 3-2-2.

**Lemma 3-2-5.** The collection of the situations  $(\heartsuit_j)$  for all  $j = 1, \dots, l$  implies the situation ( $\heartsuit$ ).

*Proof.*

We prove that the collection of the situations  $(\heartsuit_{j,fan})$  implies the following situation  $(\heartsuit_{fan})$ , which is equivalent to  $(\heartsuit)$  but in terms of the geometry of the fan  $\tilde{\sigma}$ :

( $\heartsuit_{fan}$ ) {

- For any cone  $\sigma_Z \in \tilde{\sigma}$ ,
- we have  $\sigma_Z = \sum_j \langle v_j \rangle \oplus \tau_{Z'}$  as cones,
- with the indices varying among those  $j \notin S$  with  $v_j \in \sigma_Z$
- where the cone  $\tau_{Z'} = \langle w'_k s \rangle$  is generated by
- the extremal rays  $w_k$ 's (other than those  $v_j$ 's with  $i \notin S$ ) of  $\sigma_Z$
- and is contained in a linear space  $L$  (of codimension  $\#\{j; j \in S, v_j \in \sigma_Z\}$ )
- which also contains  $a \in N$
- and where we have the exact sequence
- $0 \rightarrow L \cap N \rightarrow N \rightarrow \mathbb{Z}^{\oplus \#\{j; j \notin S, v_j \in \sigma_Z\}} \rightarrow 0$
- with the image of  $\{v_j; j \notin S, v_j \in \sigma_Z\} \in N$
- forming a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^{\oplus \#\{j; j \notin S, v_j \in \sigma_Z\}}$ .

Now having the exact sequences for the indices  $j \notin S$  with  $v_j \in \sigma_Z$

$$0 \rightarrow H_j \cap N \rightarrow N \rightarrow \mathbb{Z} \rightarrow 0$$

where  $v_j \in N$  maps to  $\pm 1 \in \mathbb{Z}$ , it is elementary to see that we have the exact sequence with  $L = \cap_{\{j; j \notin S, v_j \in \sigma_Z\}} \cap H_j$

$$0 \rightarrow L \cap N \rightarrow N \rightarrow \mathbb{Z}^{\oplus \#\{j; j \notin S, v_j \in \sigma_Z\}} \rightarrow 0$$

such that  $\{v_j; j \notin S, v_j \in \sigma_Z\}$  maps to a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^{\oplus \#\{j: j \notin S, v_j \in \sigma_Z\}}$ .

This completes the proof for Lemma 3-2-5.

Therefore, in order to achieve the situation  $(\heartsuit)$ , we only have to verify the collection of the situations  $(\heartsuit_j)$  for  $j = 1, \dots, l$ , which should follow from taking the blowup  $\mu_X$  to be the “collection” of the blowups of the  $I_{\alpha_j, X}$  for  $j = 1, \dots, l$ . This motivates us to define the torific ideal  $I$  to be the product of these:

$$I := \prod_{j=1}^l I_{\alpha_j, X}.$$

(Actually in Definition 3-2-7 we may multiply some more  $\alpha$ -torific ideals with  $\alpha \neq 1, \dots, l$ . See the definition of the torific ideal for the quasi-elementary cobordism  $B$ .)

This finishes the discussion of the torification principle for toric varieties.

Now we present the verification of the torifying property of the torific ideal according the torification principle.

**Lemma 3-2-6.** *There exists a finite set  $\mathfrak{C}$  of integers so that at each point  $q \in B$  it contains all the characters of  $G_q$  acting on the tangent space  $T_{B,q}$  (modulo  $b_q$ ).*

*Proof.*

In the case of  $B = X$  being a affine nonsingular toric variety, the requirement is equivalent to  $\{\alpha_1, \dots, \alpha_l\} \subset \mathfrak{C}$  in the notation of Observation 3-2-1. The claim for the case of a general quasi-elementary cobordism  $B$  follows from this and the fact that  $B$  can be covered by a finite number of Luna’s locally toric charts.

**Definition 3-2-7.** *Let  $\mathfrak{C}$  be a finite set of integers as above. We define the torific ideal  $I_{\mathfrak{C}}$  with respect to  $\mathfrak{C}$  to be*

$$I = \prod_{\alpha \in \mathfrak{C}} I_{\alpha}.$$

*Note that the torific ideal  $I_{\mathfrak{C}}$  does depend on the choice of  $\mathfrak{C}$  but we usually suppress this, call it the torific ideal and denote it by  $I$  without the subscript.*

**Theorem 3-2-8.** *Let  $X$  be an affine nonsingular toric variety with  $K^*$  acting as a one-parameter subgroup. Let*

$$\mu_X : X^{tor} \rightarrow X$$

*be the normalization of the blowup of the torific ideal  $I_X$  for  $X$  and  $D^{tor}$  the divisor defined by the principal ideal  $\mu_X^{-1}(I_X) \cdot \mathcal{O}_{X^{tor}}$ . Then*

*(i)  $(U_{X^{tor}} := X^{tor} - D^{tor}, X^{tor})$  has a toroidal structure with respect to which the induced  $K^*$ -action is toroidal. More precisely, we have the situation  $(\heartsuit)$  for  $X^{tor}$ .*

*(ii)  $X^{tor}$  is a quasi-elementary cobordism.*

*(iii) Denoting by  $\{\sigma_{Z_{max}}\}$  the set of the maximal cones of  $\sigma^{tor}$ , where  $\sigma^{tor}$  is the fan corresponding to the toric variety  $X^{tor} = X(N, \sigma^{tor})$ . Then each of the affine toric varieties  $\{X(N, \sigma_{Z_{max}})\}$  satisfies the condition  $(*)$ :*

(\*) If a  $K^*$ -orbit  $O(q)$  lies in  $X(N, \sigma_{Z_{max}})$ , then its closure  $\overline{O(q)}$  also lies in  $X(N, \sigma_{Z_{max}})$ .

As  $\{X(N, \sigma_{Z_{max}})\}$  cover  $X^{tor} = X(N, \sigma^{tor})$ , the quotients  $\{X(N, \sigma_{Z_{max}})/K^* = X(\pi(N), \pi(\sigma_{Z_{max}}))\}$  cover  $X^{tor}/K^*$ . Moreover, we have the commutative diagram of Cartesian products

$$\begin{array}{ccc}
 X(\pi(N), \pi(\partial_{-\sigma_{Z_{max}}})) & & X(\pi(N), \pi(\partial_{+\sigma_{Z_{max}}})) \\
 \searrow & & \swarrow \\
 \downarrow & X(\pi(N), \pi(\sigma_{Z_{max}})) & \downarrow \\
 (X^{tor})_-/K^* & & (X^{tor})_+/K^* \\
 \searrow & \downarrow & \swarrow \\
 & X^{tor}/K^*. &
 \end{array}$$

*Proof.*

We use the same notation as in Observation 3-2-1. By Lemma 3-2-4 and Lemma 3-2-5 we only have to show the situations  $(\heartsuit_j)$  for  $j = 1, \dots, l$ .

First note that  $\mu_X$  is characterized by the following properties:

- (o)  $\mu_X$  is proper birational,
- (i)  $X^{tor}$  is normal,
- (ii)  $\mu_X^{-1}(I_{\alpha, X}) \cdot \mathcal{O}_{X^{tor}}$  is principal for each  $\alpha \in S$ , and
- (iii) for any  $\mu' : X' \rightarrow X$  satisfying the properties (o), (i) and (ii), there exists a unique  $\nu' : X' \rightarrow X^{tor}$  with  $\mu' = \mu_X \circ \nu'$ .

Fix  $j = 1, \dots, l$ .

First take

$$\mu_j : X_j \rightarrow X$$

to be the normalization of the blowup of  $X$  along  $I_{\alpha_j, X}$ . Secondly, let

$$\tau_j : X^{tor} \rightarrow X_j$$

to be the normalization of the blowup of  $X_j$  along the ideal

$$\prod_{\alpha \in S, \alpha \neq \alpha_j} \mu_i^{-1}(I_{\alpha, X}) \cdot \mathcal{O}_{X_j}.$$

Then by the characterization above, we have

$$\mu_X = \mu_j \circ \tau_j.$$

We choose monomial generators of  $K^*$ -character  $\alpha_j$  for  $I_{\alpha_j, X}$  to be

$$v_j^*, m_1, \dots, m_s$$

where  $m_1, \dots, m_s$  do not contain  $v_j^*$ , i.e.,

$$m_1, \dots, m_s \in v_j^\perp \cap \sigma^\vee = \langle v_1^*, \dots, \overset{\vee}{v_j^*}, \dots, v_l^*, \pm v_{l+1}^*, \dots, v_n^* \rangle.$$

Case:  $v_j^*$  generates  $I_{\alpha_j, X}$  (and hence we do not have any of  $m_1, \dots, m_s$ ).

In this case, we have

$$D_{v_j} \subset D^{tor}.$$

Case:  $v_j^*$  does not generate  $I_{\alpha_j, X}$ .

In this case,  $X_j$  is covered by affine toric varieties which are the normalization of the affine charts of the following types for the blowup of  $X$  along  $I_{\alpha_j, X}$ .

- the chart obtained by inverting  $v_j^*$ : Over this chart, there is NO intersection with  $D_{v_j}$ .

- the chart(s) obtained by inverting, say  $m_1$ : The affine coordinate ring for the chart (before normalization) is

$$K[v_j^* - m_1, m_2 - m_1, \dots, m_s - m_1, v_1^*, \dots, \overset{\vee}{v_j^*}, \dots, v_l^*, \pm v_{l+1}^*, \dots, \pm v_n^*]$$

where

$$m_2 - m_1, \dots, m_s - m_1, v_1^*, \dots, \overset{\vee}{v_j^*}, \dots, v_l^*, \pm v_{l+1}^*, \dots, \pm v_n^* \in v_j^\perp \cap M.$$

Therefore, the normalization has the form

$$K[v_j^* - m_1] \otimes K[\{y'_{h'}\}]$$

where the monomials  $\{y'_{h'}\}$  are taken from  $v_j^\perp \cap M$ .

**Lemma 3-2-9.** *Over this chart,*

$$\mu_j^{-1}(I_{\alpha, X}) \cdot \mathcal{O}_{X_j} \quad \forall \alpha \in S, \alpha \neq \alpha_j$$

*is generated by monomials in  $v_j^\perp \cap M$ .*

*Proof.*

Let  $m \in I_{\alpha, X}$  be a monomial in  $v_1^*, \dots, v_l^*, \pm v_{l+1}^*, \dots, \pm v_n^*$  of  $K^*$ -character  $\alpha$ . Write

$$\begin{aligned} m &= cv_j^* + m' \\ &= c(v_j^* - m_1) + cm_1 + m' \end{aligned}$$

where  $c \in \mathbb{Z}_{\geq 0}$  and  $m'$  is a monomial in  $v_j^\perp \cap M$ . Since  $v_j^* - m_1$  has the trivial  $K^*$ -character,  $cm_1 + m'$  has the  $K^*$ -character  $\alpha$  and it is a monomial in  $v_j^\perp \cap M$ . As these monomials  $\{m\}$  generate  $\mu_j^{-1}(I_{\alpha, X})$ , so do  $\{cm_1 + m'\}$  and we have the claim.

By the above lemma, over this chart  $X^{tor}$ , which is obtained as the normalization of the blowup of  $X_j$  along  $\prod_{\alpha \in S, \alpha \neq \alpha_j} \mu_j^{-1}(I_{\alpha, X}) \cdot \mathcal{O}_{X_j}$ , is covered by the affine toric charts of the form

$$\text{Spec}\{K[v_j^* - m_1] \otimes K[\{y_h\}]\}$$

where the monomials  $\{y_h\}$  are taken from  $v_j^\perp \cap M$  and where in the exact sequence

$$0 \rightarrow v_j^\perp \cap M \rightarrow M \rightarrow \mathbb{Z} \rightarrow 0$$

the image of  $v_j^* - m_1 \in M$  is  $\pm 1 \in \mathbb{Z}$ . These affine charts satisfy the condition in the situation  $(\heartsuit_j)$ .

Varying the charts (i.e., inverting the monomial generators  $m_1, \dots, m_s$ ), we see that the fan  $\sigma^{tor}$ , where  $X^{tor} = X(N, \sigma^{tor})$ , is covered by the cones satisfying the condition in the situation  $(\heartsuit_{j, fan})$ . Therefore, we conclude that any cone in  $\sigma^{tor}$  satisfies the condition in the situation  $(\heartsuit_j)$ , that is to say, we have the situation  $(\heartsuit_j)$ .

This concludes the proof of the assertion (i) of Theorem 3-2-8.

Now we verify the assertion (ii) that  $X^{tor}$  is a quasi-elementary cobordism.

Suppose, on the contrary, that there is a point  $q \in X^{tor}$  such that  $q$  is not a fixed point  $q \notin F_{X^{tor}}$  and that both limits  $\lim_{t \rightarrow \infty} t(q)$  and  $\lim_{t \rightarrow 0} t(q)$  exists in  $X^{tor}$ .

Then it is necessarily the case that  $\mu_X(q) \in X$  is not a fixed point.

In fact, consider the natural finite morphism

$$f : X^{tor} \rightarrow X \times \mathbb{P}^N$$

where the  $N + 1$  homogeneous coordinates  $(f_0, \dots, f_N)$  of  $\mathbb{P}^N$  are given by the monomial generators of the toric ideal  $I_X$  of the same  $K^*$ -character  $\prod_{\alpha \in S} \alpha$ . If  $\mu_X(q)$  is a fixed point, then  $f(q) = \mu_X(q) \times (f_0(q), \dots, f_N(q))$  is also a fixed point in  $X \times \mathbb{P}^N$ . As  $f$  is a finite morphism, this would imply that  $q$  is a fixed point, a contradiction.

Since  $\mu_X$  is proper, both limits  $\lim_{t \rightarrow \infty} t(\mu_X(q)) = \mu_X(\lim_{t \rightarrow \infty} t(q))$  and  $\lim_{t \rightarrow 0} t(\mu_X(q)) = \mu_X(\lim_{t \rightarrow 0} t(q))$  exist in  $X$ , implying that  $X$  is not quasi-elementary. This is a contradiction, as  $X$  is affine and hence quasi-elementary.

Finally we verify the assertion (iii).

Observe that  $q \in X^{tor}$ , not being a fixed point, has the limit  $\lim_{t \rightarrow \infty} t(q)$  in  $X^{tor}$  (resp.  $\lim_{t \rightarrow 0} t(q)$ ) if and only if for the cone  $\sigma_Z$  whose corresponding orbit  $O(\sigma_Z)$  contains  $q$  we have a cone  $(\sigma_Z)^\infty$  (resp.  $(\sigma_Z)^0$ ) such that there exists a point  $x \in \sigma_Z$  with  $x + \epsilon a \in (\sigma_Z)^\infty$  (resp. for all  $x - \epsilon a \in (\sigma_Z)^0$  sufficiently small positive number  $\epsilon$ ). From this observation and from the fact that  $X^{tor}$  is quasi-elementary, it follows that  $X(N, \sigma_{Z_{max}})$  satisfies the condition (\*) for each of the maximal cones  $\sigma_{Z_{max}}$  in  $\sigma^{tor}$ . Thus we have the inclusion

$$X(N, \sigma_{Z_{max}})/K^* \hookrightarrow X^{tor}/K^*$$

and the commutative diagram follows from the main example.

This completes the proof of Theorem 3-2-8.

**Remark 3-2-10.**

(i) By the argument for the assertion (iii) we see that  $X^{tor}$  being quasi-elementary is equivalent to the condition that NO two maximal cones  $\sigma_{Z_{1,max}}$  and  $\sigma_{Z_{2,max}}$  with one being on top of the other: there exists a point  $x \in \sigma_{Z_{1,max}} \cap \sigma_{Z_{2,max}}$  such that  $x + \epsilon a \in \sigma_{Z_{1,max}}$  and  $x - \epsilon a \in \sigma_{Z_{2,max}}$  for all sufficiently small positive number  $\epsilon$ .

This in turn is equivalent to saying that the subdivision of  $\sigma$  to obtain  $\sigma^{tor}$  “comes from downstairs”: Let  $\pi : N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}/\mathbb{R} \cdot a$  be the projection. Then there is a fan  $\pi(\sigma)^{tor}$  obtained by subdividing  $\pi(\sigma)$  so that the fan  $\sigma^{tor}$  is covered by the cones  $\{\pi^{-1}(\tau) \cap \sigma; \tau \in \pi(\sigma)^{tor}\}$ .

The last condition follows easily from the observation below, giving an alternative interpretation for the assertion (iii):

If  $f_0, \dots, f_N$  are the monomial generators of the toric ideal  $I_X$ , then the blowup of  $X$  along  $I_X$  is covered by the affine charts of the form (say, inverting  $f_0$ )

$$\text{Spec } K[\sigma^{\vee} \cap M][\frac{f_1}{f_0}, \dots, \frac{f_N}{f_0}]$$

where  $\frac{f_1}{f_0}, \dots, \frac{f_N}{f_0}$  all have the trivial  $K^*$ -character, i.e., they are all  $K^*$ -invariants.

(ii) For a special choice of  $S$  so that it contains a nonzero character  $\beta$  which is divisible by all  $\alpha_j$  for  $j = 1, \dots, l$  we can show that

$$\pi(\partial_{-}\sigma^{tor}) = \pi(\partial_{+}\sigma^{tor})$$

and hence the two toroidal embeddings, obtained by excluding those coordinate divisors  $D_{\pi(v_i)}$  with  $i \notin S$  from the boundaries, also coincide

$$(U_{X^{tor}}, X^{tor})_-/K^* = (U_{X^{tor}}, X^{tor})_+/K^*.$$

As we do not use this fact in our proof, we only refer the reader to Abramovich-Karu-Matsuki-Włodarczyk [1] for a proof.

**Example 3-2-11.**

Here we give an easy example to demonstrate the torification principle for toric varieties.

Take  $X = X(N, \sigma) = \mathbb{A}^3$  where  $\sigma$  is the regular cone

$$\sigma = \langle v_1, v_2, v_3 \rangle \subset N_{\mathbb{R}}$$

and where  $t \in K^*$  acts on the coordinates as

$$t(z_1, z_2, z_3) = (t^2 z_1, t z_2, t^{-1} z_3),$$

i.e., it corresponds to

$$a = (2, 1, -1) \in N.$$

We have the following monomial generators of the  $\alpha$ -toric ideals for  $\alpha = 2, 1, -1$

$$\begin{aligned} I_{2,X} &= \{z_1, z_1^2\} \\ I_{1,X} &= \{z_2, z_1 z_3\} \\ I_{-1,X} &= \{z\}. \end{aligned}$$

Set the torific ideal  $I_X$  (with respect to the set  $\mathfrak{C} = \{2, 1, -1\}$ ) to be

$$I_X = I_{2,X} I_{1,X} I_{-1,X}.$$

Then the normalization  $(\mathbb{A}^3)^{tor} = X(N, \sigma^{tor})$  of the blowup of  $X$  along the torific ideal  $I_X$  is described by the fan  $\sigma^{tor}$  consisting of the following three maximal cones

$$\begin{aligned}\sigma_1 &= \langle v_1, v_3, 2v_1 + v_2 \rangle \\ \sigma_2 &= \langle v_2, v_1 + v_2, v_2 + v_3 \rangle \\ \sigma_3 &= \langle v_3, 2v_1 + v_2, v_1 + v_2, v_2 + v_3 \rangle.\end{aligned}$$

We have

$$D^{tor} = D_{v_3} \cup D_{v_1+v_2} \cup D_{v_2+v_3} \cup D_{2v_1+v_2}.$$

Thus the toroidal structure  $(U_{X^{tor}}, X^{tor})$  obtained from

$$(T_{X^{tor}} = X^{tor} - \{(D_{v_1} \cup D_{v_2}) \cup (D_{v_3} \cup D_{v_1+v_2} \cup D_{v_2+v_3} \cup D_{2v_1+v_2})\}, X^{tor})$$

by removing  $D_{v_1}$  and  $D_{v_2}$  from the boundary divisor. In fact, we observe the following.

The cone  $\sigma_1$  has the product structure

$$\sigma_1 = \langle v_1 \rangle \oplus \langle v_3, 2v_1 + v_2 \rangle,$$

from which we have

$$X(N, \sigma_1) = \text{Spec}\{K[v_1^* - 2v_2^*] \otimes K[v_2^*, v_3^*]\} \cong \mathbb{A}_1 \times Z'.$$

with  $D_{v_1} \cap X(N, \sigma_1)$  defined by  $\{v_1^* - 2v_2^* = 0\}$ .

Thus we see that even after removing  $D_{v_1}$  from the boundary divisor

$$(U_{X^{tor}} \cap X(N, \sigma_1), X(N, \sigma_1)) = (\mathbb{A}_1 - \{0\}, \mathbb{A}_1) \times (T_{Z'}, Z') \cup (\mathbb{A}_1 - \{1\}, \mathbb{A}_1) \times (T_{Z'}, Z')$$

has the toroidal structure with respect to which  $K^*$ -action is toroidal.

The cone  $\sigma_2$  has the product structure

$$\sigma_2 = \langle v_2 \rangle \oplus \langle v_1 + v_2, v_2 + v_3 \rangle,$$

from which we have

$$X(N, \sigma_2) = \text{Spec}\{K[v_2^* - (v_1^* + v_3^*)] \otimes K[v_1^*, v_3^*]\} \cong \mathbb{A}_1 \times Z''.$$

with  $D_{v_2} \cap X(N, \sigma_2)$  defined by  $\{v_2^* - (v_1^* + v_3^*) = 0\}$ .

Thus we see that even after removing  $D_{v_2}$  from the boundary divisor

$$(U_{X^{tor}} \cap X(N, \sigma_2), X(N, \sigma_2)) = (\mathbb{A}_1 - \{0\}, \mathbb{A}_1) \times (T_{Z''}, Z'') \cup (\mathbb{A}_1 - \{1\}, \mathbb{A}_1) \times (T_{Z''}, Z'')$$

has the toroidal structure with respect to which  $K^*$ -action is toroidal.

The cone  $\sigma_3$  does not contain  $v_1$  or  $v_2$  and hence

$$(U_{X^{tor}} \cap X(N, \sigma_3), X(N, \sigma_3)) = (T_{X(N, \sigma_3)}, X(N, \sigma_3))$$

has the original toric and hence structure with respect to which the  $K^*$ -action is toric and hence toroidal.

Once the torification principle for toric varieties is established, it is straightforward to verify the torifying property of the torific ideal for a quasi-elementary cobordism via Luna's locally toric charts.

**Theorem 3-2-12.** Let  $B$  be a nonsingular quasi-elementary cobordism with  $K^*$  acting effectively. Let

$$\mu : B^{tor} \rightarrow B$$

be the normalization of the blowup of  $B$  along the torific ideal  $I$  and  $D^{tor}$  the divisor defined by the principal ideal  $\mu^{-1}(I) \cdot \mathcal{O}_B$ . Then

$$(U_{B^{tor}} = B^{tor} - D^{tor}, B^{tor})$$

has a toroidal structure with respect to which the induced  $K^*$ -action is toroidal.

Moreover,  $B^{tor}$  is a quasi-elementary cobordism, which induces a  $V$ -toroidal birational map  $\varphi$  as below:

$$\begin{array}{ccc} (U_{B^{tor}}, B^{tor})_- / K^* & \xrightarrow{\varphi} & (U_{B^{tor}}, B^{tor})_+ / K^* \\ \searrow & & \swarrow \\ (U_{B^{tor}}, B^{tor}) / K^* & & \end{array}$$

*Proof.*

We take Luna's locally toric chart for  $p \in B - F^*$

$$X \xleftarrow{\eta} V \xrightarrow{i} Y = U_p \subset B$$

where  $\eta$  and  $i$  are strongly étale with  $i$  being surjective and where  $Y = U_p$  satisfies the condition (\*). (Note that the collection  $\{U_p; p \in B - F^*\}$  provides an open covering of  $B$ .)

We use the same notation as in the proof of Claim 3-1-6.

Since for any point  $q \in Y - F_Y^* = Y \cap (B - F^*)$  (which implies  $q_V \in V - F_V^*$  and  $q_X \in X - F_X^*$ ), we have

$$\widehat{(\eta^* I_{\alpha, X})}_{q_V} = \widehat{(I_{\alpha, V})}_{q_V} = \widehat{(i^* I_\alpha)}_{q_V}$$

with all sides being characterized as the ideal in  $\widehat{\mathcal{O}_{V, q_V}}$  generated by all the semi-invariant functions of  $G_{q_V}$  ( $= G_q = G_{q_X}$ )-character  $\alpha$  (See Lemma 3-1-7.), which implies

$$(\eta^* I_{\alpha, X})_{q_V} = (I_{\alpha, V})_{q_V} = (i^* I_\alpha)_{q_V}.$$

Since  $\eta^* I_{\alpha, X}, I_{\alpha, V}$  and  $i^* I_\alpha$  are  $K^*$ -equivariant, this implies that the above equality holds not only for  $q \in Y - F_Y^*$  but also for all  $q \in Y$ , i.e.,

$$\eta^* I_{\alpha, X} = I_{\alpha, V} = i^* I_\alpha.$$

Thus we conclude that the torific ideal on  $X$  pulls back to the torific ideal on  $V$ , coinciding with the torific ideal on  $B$  pulled back to  $V$ , i.e.,

$$\eta^* I_X = I_V = i^* I.$$

This gives rise to the commutative diagram of Cartesian products

$$\begin{array}{ccccccc} (U_{X^{tor}}, X^{tor}) & \xleftarrow{\eta^{tor}} & (U_{V^{tor}}, V^{tor}) & \xrightarrow{i^{tor}} & (U_{Y^{tor}}, Y^{tor}) & \subset & (U_{B^{tor}}, B^{tor}) \\ \mu_X \downarrow & & \mu_V \downarrow & & \mu_Y \downarrow & & \\ X & \xleftarrow{\eta} & V & \xrightarrow{i} & Y & \subset & B. \end{array}$$

For any point  $r \in Y^{tor}$ , we can find by the torification principle for the toric variety  $X$  an affine open set  $Z_{\prod\{\delta_j\}}$  such that

$$r_{X^{tor}} \in (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) := Z_{\prod\{\delta_j\}} \cap (U_{X^{tor}}, X^{tor}) \cong \prod(\mathbb{A}^1 - \{\delta_j\}, \mathbb{A}^1 - \{\delta_j\}) \times (T_{Z'}, Z')$$

where  $Z'$  is an affine toric variety with the torus  $T_{Z'}$  and the  $\delta_i$  are either 0 or 1.

In order to obtain Luna's toroidal chart for  $q \in Y^{tor} \subset B^{tor}$

$$(U_{X_r}, X_r) \leftarrow (U_{V_r}, V_r) \rightarrow (U_{Y_r}, Y_r) \subset (U_{Y^{tor}}, Y^{tor}) \subset (U_{B^{tor}}, B^{tor}),$$

we set

$$\begin{aligned} (U_{X_r}, X_r) &:= (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) \\ (U_{V_r}, V_r) &:= (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) \times_{X//K^*} V//K^* \\ (U_{Y_r}, Y_r) &:= (U_{V_r}, V_r) \end{aligned}$$

when  $p \in Y$  is a fixed point and hence  $i$  and  $i^{tor}$  are isomorphisms by construction of Luna's locally toric charts, and

$$\begin{aligned} (U_{X_r}, X_r) &:= (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) \\ (U_{V_r}, V_r) &:= (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) \times_{X//K^*} V//K^* \cap (i^{tor})^{-1}(Y_r) \\ (U_{Y_r}, Y_r) &:= Y_r \cap (U_Y, Y) \end{aligned}$$

where  $r \in Y_r$  is a small affine open neighborhood so that  $Y_r \subset i^{tor}(V_r)$  when  $p \in Y$  is not a fixed point and hence all the orbits in  $Y, V, X$  and  $Y^{tor}, V^{tor}, X^{tor}$  are closed by construction of Luna's locally toric charts.

Therefore,  $(U_{B^{tor}}, B^{tor})$  has a toroidal structure with respect to which the induced  $K^*$ -action is toroidal.

It remains to prove that  $B^{tor}$  is a quasi-elementary cobordism.

Suppose that  $B^{tor}$  is not quasi-elementary, i.e., there is a point  $q \in B^{tor}$  such that  $q$  is not a fixed point and that both limits  $\lim_{t \rightarrow \infty} t(q)$  and  $\lim_{t \rightarrow 0} t(q)$  exist in  $B^{tor}$ . Take  $p \in B - F^*$  so that  $Y = U_p$  (in Luna's locally toric chart for  $p$ ) contains  $\mu(q)$ . Note that since  $U$  satisfies the condition (\*),  $Y^{tor} = U^{tor} \subset B^{tor}$  also satisfies the condition (\*). In particular, both limits sit inside of  $Y^{tor}$ . On the other hand, since both  $\eta^{tor}$  and  $i^{tor}$  are strongly étale as so are  $\eta$  and  $i$ , this implies that  $q_{X^{tor}} \in X^{tor}$  is not a fixed point and that both limits  $\lim_{t \rightarrow \infty} t(q_{X^{tor}})$  and  $\lim_{t \rightarrow 0} t(q_{X^{tor}})$  exist in  $X^{tor}$ . This contradicts the fact by Theorem 3-2-8 that  $X^{tor}$  is quasi-elementary.

The commutative diagram indicating  $\varphi$  is  $V$ -toroidal is an immediate consequence of the toroidal structure  $(U_{B^{tor}}, B^{tor})$  (with respect to which the induced  $K^*$ -action is toroidal) and  $B^{tor}$  being quasi-elementary.

This completes the proof of Theorem 3-2-12.

## CHAPTER 4. RECOVERY OF NONSINGULARITY

Let us recall what we have established so far.

In Chapter 2, we constructed a birational cobordism  $B$  for a given birational map  $\phi : X_1 \dashrightarrow X_2$  (after replacing it with some projective birational morphism by eliminating points of indeterminacy by sequences of blowups with smooth centers). The birational cobordism gives rise to the decomposition of  $\phi$  into a sequence of (V-)locally toric birational maps

$$\phi : X_1 = W_1 \xrightarrow{\varphi_1} W_2 \xrightarrow{\varphi_2} \cdots W_i \xrightarrow{\varphi_i} W_{i+1} \cdots \xrightarrow{\varphi_{s-2}} W_{s-2} \xrightarrow{\varphi_{s-1}} W_s = X_2$$

where each  $\varphi_i$  corresponds to the commutative diagram

$$\begin{array}{ccc} W_i & \xrightarrow{\varphi_i} & W_{i+1} \\ \| & & \| \\ (B_{a_i})_-/K^* & & (B_{a_i})_+/K^* \\ \searrow & & \swarrow \\ & (B_{a_i})/K^* & \end{array}$$

induced by the quasi-elementary cobordism  $B_{a_i}$ , associated to the value  $a_i$  of the strictly increasing function from the set of the connected components of the fixed point set of  $B$  to their characters in  $\mathbb{Z}$ .

The main object of Chapter 3 was to transform locally toric birational maps into toroidal birational transformations. By blowing up the toric ideal, we obtain a toroidal structure  $(U_{B_{a_i}^{tor}}, B_{a_i}^{tor})$  with respect to which the  $K^*$ -action is toroidal and hence which induces the toroidal birational map  $\varphi_i^{tor}$  as in the commutative diagram below

$$\begin{array}{ccc} (U_{W_{i-}^{tor}}, W_{i-}^{tor}) & \xrightarrow{\varphi_i^{tor}} & (U_{W_{i+}^{tor}}, W_{i+}^{tor}) \\ \| & & \| \\ (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_-/K^* & & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_+/K^* \\ \searrow & & \swarrow \\ & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^* & \end{array}$$

Since the blowup morphism  $\mu : B_{a_i}^{tor} \rightarrow B_{a_i}$  gives rise to a commutative diagram

$$\begin{array}{ccc}
(U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_-/K^* & & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_+/K^* \\
\searrow & & \swarrow \\
\downarrow & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^* & \downarrow \\
(B_{a_i})_-/K^* & & (B_{a_i})_+/K^* \\
\searrow & \downarrow & \swarrow \\
& (B_{a_i})/K^* &
\end{array}$$

we actually factor  $\varphi_i$  into a sequence of birational maps

$$\varphi_i : W_i \xleftarrow{f_{i-}} (U_{W_{i-}^{tor}}, W_{i-}^{tor}) \dashrightarrow (U_{W_{i+}^{tor}}, W_{i+}^{tor}) \xrightarrow{f_{i+}} W_{i+1}.$$

If  $W_i, W_{i-}^{tor}, W_{i+}^{tor}$  and  $W_{i+1}$  were all nonsingular and both  $f_{i-}$  and  $f_{i+}$  were sequences of blowups with smooth centers, then we would only have to apply the strong factorization theorem of Morelli [1,2] or Abramovich-Matsuki-Rashid] [1] (cf.Włodarczyk [1]) to the toroidal birational map  $\varphi_i^{tor}$  between nonsingular toroidal embeddings to finish the proof of the weak factorization theorem. But in general these varieties are all singular, while  $f_{i-}$  and  $f_{i+}$  are not sequences of blowups with smooth centers.

The purpose of Chapter 4 is to bring this SINGULAR situation back to a NON-SINGULAR one by an application of the canonical resolution of singularities.

More precisely, we construct the following commutative diagram

$$\begin{array}{ccccc}
W_i^{res} & \xleftarrow{p_{i-}} & W_{i-}^{can} & \xrightarrow{\varphi_i^{can}} & W_{i+}^{can} & \xrightarrow{p_{i+}} & W_{i+1}^{res} \\
r_{i-} \downarrow & & h_{i-} \downarrow & & \downarrow h_{i+} & & \downarrow r_{i+} \\
W_i & \xleftarrow{f_{i-}} & W_{i-}^{tor} & \xrightarrow{\varphi_i^{tor}} & W_{i+}^{tor} & \xrightarrow{f_{i+}} & W_{i+1}
\end{array}$$

where

- (i)  $W_i^{res}, W_{i-}^{can}, W_{i+}^{can}$  and  $W_{i+1}^{res}$  are all nonsingular,
- (ii) the morphisms  $p_{i-}$  and  $p_{i+}$  are sequences of blowups with smooth centers, and
- (iii) by setting

$$\begin{aligned}
U_{W_{i-}^{can}} &= h_{i-}^{-1}(U_{W_{i-}^{tor}}) \\
U_{W_{i+}^{can}} &= h_{i+}^{-1}(U_{W_{i+}^{tor}})
\end{aligned}$$

the birational map

$$\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$$

is V-toroidal via the commutative diagram

$$\begin{array}{ccc}
(U_{W_{i-}^{can}}, W_{i-}^{can}) & \xrightarrow{\varphi_i^{can}} & (U_{W_{i+}^{can}}, W_{i+}^{can}) \\
h_{i-} \downarrow & & \downarrow h_{i+} \\
(U_{W_{i-}^{tor}}, W_{i-}^{tor}) & \xrightarrow{\varphi_i^{tor}} & (U_{W_{i+}^{can}}, W_{i+}^{tor}) \\
& \searrow & \swarrow \\
& (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^*. &
\end{array}$$

After the construction, we only have to apply the strong factorization theorem of toroidal birational maps to  $\varphi_i^{can}$  to finish the proof of the weak factorization theorem.

#### §4-1. Application of the canonical resolution of singularities

##### Remark 4-1-1.

(i) By the **canonical resolution of singularities**, we mean an algorithm which, given a variety  $X$ , provides a uniquely determined sequence of blowups with smooth centers  $r : X^{res} \rightarrow X$  from a nonsingular variety, satisfying the conditions:

( $\spadesuit^{res} - 0$ ) The centers of blowups are taken only over the singular locus  $Sing(X)$ .

( $\spadesuit^{res} - 1$ ) If  $Y \rightarrow X$  is a smooth morphism of varieties, then the canonical resolution of  $Y$  is obtained by pulling back that of  $X$ , i.e., we have a Cartesian product

$$\begin{array}{ccc}
Y & \longleftarrow & Y \times_X X^{res} = Y^{res} \\
\downarrow & & \downarrow \\
X & \longleftarrow & X^{res}.
\end{array}$$

In particular, the canonical resolution can be pulled back by an open immersion or an étale morphism.

The condition ( $\spadesuit^{res} - 1$ ) also implies that any automorphism of  $X$  (not necessarily over  $K$ ) lifts to that of the canonical resolution  $X^{res}$ .

Moreover, the condition ( $\spadesuit^{res} - 1$ ) implies that a family of automorphisms  $\theta_g$  of a variety  $X$  parametrized by a smooth variety  $g \in G$  lifts to the family of automorphisms of the canonical resolution  $X^{res}$  (for example an action of an algebraic group  $G$  on a variety  $X$ ). In fact, applying the condition ( $\spadesuit^{res} - 1$ ) to the map parametrizing the automorphisms of  $X$

$$\theta : G \times X \rightarrow X \text{ given by } \theta(g, x) = \theta_g(x)$$

and applying it also to the projection onto the second factor  $pr_2 : G \times X \rightarrow X$  we obtain the map parametrizing the automorphisms of  $X^{res}$

$$\theta^{res} : G \times X^{res} = (G \times X) \times_{pr_2 X} X^{res} = (G \times X) \times_{\theta X} X^{res} = Y^{res} \rightarrow X^{res}.$$

(ii) By the **canonical principalization of ideals**, we mean an algorithm which, given an ideal sheaf  $I$  on a nonsingular variety  $X$ , provides a uniquely determined sequence of blowups with smooth (and admissible) centers  $p : X^{can} \rightarrow X$  from a

nonsingular variety with the property  $p^{-1}(I) \cdot \mathcal{O}_{X^{can}}$  being principal, satisfying the conditions:

( $\spadesuit^{can} - 0$ ) The centers of blowups are taken over the support of  $\mathcal{O}_X/I$ .

( $\spadesuit^{can} - 1$ ) If  $f : Y \rightarrow X$  is a smooth morphism of nonsingular varieties (not necessarily over the base field  $K$ ), then the canonical principalization of the ideal sheaf  $f^*I$  is obtained by pulling back that of  $I$ , i.e., we have a Cartesian product

$$\begin{array}{ccc} Y & \longleftarrow & Y \times_X X^{can} = Y^{can} \\ \downarrow & & \downarrow \\ X & \longleftarrow & X^{can}. \end{array}$$

In particular, the canonical principalization (of the corresponding ideals) can be pulled back by an open immersion or an étale morphism.

Also the condition ( $\spadesuit^{can} - 1$ ) implies that any automorphism of  $X$  stabilizing the ideal  $I$  (not necessarily over  $K$ ) lifts to that of the canonical principalization  $X^{can}$ .

Moreover, the condition ( $\spadesuit^{can} - 1$ ) implies that a family of automorphisms  $\theta_g$  of a variety  $X$  parametrized by a smooth variety  $g \in G$  lifts to the family of automorphisms of the canonical principalization  $X^{can}$  (for example an action of an algebraic group  $G$  on a variety  $X$ ), if the ideal  $I$  is stable under the automorphisms, i.e.,  $\theta_g^*I = I \quad \forall g \in G$ . In fact, applying the condition ( $\spadesuit^{can} - 1$ ) to the map parametrizing the automorphisms of  $X$

$$\theta : G \times X \rightarrow X \text{ given by } \theta(g, x) = \theta_g(x)$$

and applying it to the projection onto the second factor  $pr_2 : G \times X \rightarrow X$  and also noting that the stability implies  $\theta^*I = pr_2^*I$ , we obtain the map parametrizing the automorphisms of  $X^{can}$  as before

$$\theta^{can} : G \times X^{can} = (G \times X) \times_{pr_2 X} X^{can} = (G \times X) \times_{\theta X} X^{can} = Y^{can} \rightarrow X^{can}.$$

(iii) All the existing algorithms for the canonical resolution of singularities and/or canonical principalization of ideals (cf. Bierstone-Milman [1] Villamayor [1] Encinas-Villamayor [1] and Hironaka [2]) satisfy the requirements in (i) and (ii) as above. As these are the only properties we utilize in our proof, one may choose any one of the algorithms and we denote the resulting varieties  $X^{res}$  and  $X^{can}$  without specifying the choice.

We go back to the construction of the commutative diagram for the purpose of recovery from singular to nonsingular.

We take  $W_i^{res}$  and  $W_{i+1}^{res}$  to be the canonical resolution of singularities of  $W_i$  and  $W_{i+1}$ , respectively.

**Lemma 4-1-2.** *The morphism  $f_{i-} : W_{i-}^{tor} \rightarrow W_i$  is a projective morphism (in the sense of Grothendieck [1] but not in the sense of Hartshorne [1]) with an effective Cartier divisor  $E_{i-}^{tor}$  such that*

(i) *the support of  $E_{i-}^{tor}$  lies in  $\{D^{tor} \cap (B_{a_i}^{tor})_-\}/K^*$ , where  $D^{tor}$  is the divisor defined by the principal ideal  $\mu^{-1}(I) \cdot \mathcal{O}_{B_{a_i}^{tor}}$  with  $\mu : B_{a_i}^{tor} \rightarrow B_{a_i}$  being the normalization of the blowup of the torific ideal  $I$ ,*

- (ii)  $\mathcal{O}_{W_{i-}^{tor}}(-E_{i-}^{tor})$  is  $f_{i-}$ -ample, and
- (iii) the morphism  $f_{i-}$  is the blowup of the ideal

$$I_{i-} = (f_{i-})_* \mathcal{O}_{W_{i-}^{tor}}(-E_{i-}^{tor}).$$

Similarly, the morphism  $f_{i+} : W_{i+}^{tor} \rightarrow W_{i+1}$  is a projective morphism with an effective Cartier divisor satisfying the conditions (i), (ii) and (iii) with the negative sign – replaced by the positive one +.

*Proof.*

Since  $\mu : B_{a_i}^{tor} \rightarrow B_{a_i}$  is the normalization of the blowup of  $B_{a_i}$  along the toric ideal  $I$ , we can take an effective Cartier divisor  $E_i^{tor}$  such that the principal ideal  $\mu^{-1}(I) \cdot \mathcal{O}_{B_{a_i}^{tor}} = \mathcal{O}_{B_{a_i}^{tor}}(-E_i^{tor})$  is  $\mu$ -ample and hence  $\mu|_{(B_{a_i}^{tor})_-}$ -ample when restricted to  $\mu|_{(B_{a_i}^{tor})_-} : (B_{a_i}^{tor})_- \rightarrow (B_{a_i})_-$ .

Now if we look at Luna's locally toric chart

$$X = X(N, \sigma) = X \xleftarrow{\eta} V \xrightarrow{i} Y = U \subset B_{a_i},$$

the morphisms

$$\begin{aligned} (B_{a_i}^{tor})_- &\rightarrow (B_{a_i})_- \\ (B_{a_i}^{tor})_-/K^* &\rightarrow (B_{a_i})_-/K^* \end{aligned}$$

correspond to the toric morphisms

$$\begin{aligned} X(N, \partial_{-}\sigma^{tor}) &\rightarrow X(N, \partial_{-}\sigma) \\ X(\pi(N), \pi(\partial_{-}\sigma^{tor})) &\rightarrow X(\pi(N), \pi(\partial_{-}\sigma)), \end{aligned}$$

where  $\pi : \mathbb{N}_{\mathbb{R}} \rightarrow \mathbb{N}_{\mathbb{R}}/a \cdot \mathbb{R}$  is the projection (cf. the notation in Proposition 3-1-8 or Observation 3-2-1).

Remark also that since the toric ideal  $I$  corresponds to the toric toric ideal  $I_X$ , i.e.,  $\eta^* I_X = i^* I$ , the Cartier divisor  $E_i^{tor}$  corresponds to the toric Cartier divisor  $(E_i^{tor})_{X^{tor}}$  with  $\eta^*(E_i^{tor})_{X^{tor}} = i^* E_i^{tor}$ .

Say, the toric Cartier divisor  $(E_i^{tor})_{X^{tor}}$  is associated to a piecewise linear function  $\psi_{(E_i^{tor})_{X^{tor}}}$  on the fan  $\sigma^{tor}$  (cf. Fulton [1] or Oda [1]). Since  $\pi : \partial_{-}\sigma^{tor} \rightarrow \pi(\partial_{-}\sigma^{tor})$  is a linear isomorphism of fans (without taking the lattices  $N$  and  $\pi(N)$  into consideration), there exists a piecewise linear function  $\psi_{(E_i^{tor})_{(X^{tor})_-}/K^*}$  on the fan  $\pi(\partial_{-}\sigma^{tor})$  such that

$$\psi_{(E_i^{tor})_{X^{tor}}|_{\partial_{-}\sigma^{tor}}} = \pi^*(\psi_{(E_i^{tor})_{(X^{tor})_-}/K^*}).$$

Take  $(E_{i-}^{tor})_{(X^{tor})_-}/K^*$  to be the effective  $\mathbb{Q}$ -Cartier divisor associated to the piecewise linear function  $\psi_{(E_{i-}^{tor})_{(X^{tor})_-}/K^*}$ . (Recall that  $(E_{i-}^{tor})_{(X^{tor})_-}/K^*$  is only guaranteed to be  $\mathbb{Q}$ -Cartier, since in general  $\pi : N|_{\partial_{-}\sigma^{tor}} \rightarrow \pi(N)|_{\pi(\partial_{-}\sigma^{tor})}$  is not an isomorphism.)

Observe the following implications:

$[\mathcal{O}_{X^{tor}}(-(E_i^{tor})_{X^{tor}}) \text{ is } \mu_X\text{-ample}]$

$\implies$

$[\mathcal{O}_{X^{tor}}(-(E_i^{tor})_{X^{tor}})|_{(X^{tor})_-} \text{ is } \mu_X|_{(X^{tor})_-}\text{-ample}]$

$\iff$

$[-\psi_{(E_i^{tor})_{X^{tor}}|_{\partial_-\sigma^{tor}}} \text{ is strictly convex when restricted to each cone in } \partial_-\sigma \text{ with respect to } \partial_-\sigma^{tor}]$

$\iff$

$[-\psi_{(E_{i-}^{tor})_{(X^{tor})_-/K^*}} \text{ is strictly convex when restricted to each cone in } \pi(\partial_-\sigma) \text{ with respect to } \pi(\partial_-\sigma^{tor})]$

$\iff$

$[\mathcal{O}_{(X^{tor})_-/K^*}(-(E_{i-}^{tor})_{(X^{tor})_-/K^*}) \text{ is } (f_{i-})_X\text{-ample where } (f_{i-})_X \text{ is the morphism } (f_{i-})_X : (X^{tor})_-/K^* \rightarrow X_-/K^*.]$

Therefore, if we take  $E_{i-}^{tor}$  to be the effective  $\mathbb{Q}$ -Cartier divisor on  $W_{i-}^{tor}$  corresponding to  $(E_{i-}^{tor})_{X^{tor}}$ , then the conditions (i) and (ii) are satisfied.

By replacing it with a sufficiently divisible and high multiple, we see that there exists an effective Cartier divisor  $E_{i-}^{tor}$  satisfying all the conditions (i), (ii) and (iii).

This completes the proof of Lemma 4-1-2.

We take  $W_{i-}^{can}$  (resp.  $W_{i+}^{can}$ ) to be canonical principalization of the ideal  $(r_{i-})^{-1}(I_{i-}) \cdot \mathcal{O}_{W_i^{res}}$  (resp.  $(r_{i+})^{-1}(I_{i+}) \cdot \mathcal{O}_{W_{i+1}^{res}}$ ).

**Proposition 4-1-3.** *Let*

$$\begin{aligned} h_{i-} &: W_{i-}^{can} \rightarrow W_{i-}^{tor} \\ h_{i+} &: W_{i+}^{can} \rightarrow W_{i+}^{tor} \end{aligned}$$

be the induced morphisms and set

$$\begin{aligned} U_{W_{i-}^{can}} &= h_{i-}^{-1}(U_{W_{i-}^{tor}}) \\ U_{W_{i+}^{can}} &= h_{i+}^{-1}(U_{W_{i+}^{tor}}). \end{aligned}$$

Then

- (i) both  $(U_{W_{i-}^{can}}, W_{i-}^{can})$  and  $(U_{W_{i+}^{can}}, W_{i+}^{can})$  have the toroidal structures,
- (ii) the induced birational map  $\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$  is  $V$ -toroidal via the commutative diagram

$$\begin{array}{ccc}
(U_{W_{i-}^{can}}, W_{i-}^{can}) & & (U_{W_{i+}^{can}}, W_{i+}^{can}) \\
\downarrow h_{i-} & & \downarrow h_{i+} \\
(U_{W_{i-}^{tor}}, W_{i-}^{tor}) & & (U_{W_{i+}^{tor}}, W_{i+}^{tor}) \\
\parallel & & \parallel \\
(U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_-/K^* & & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_+/K^* \\
\searrow & & \swarrow \\
& (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^*. &
\end{array}$$

*Proof.*

We take Luna's locally toric chart for  $p \in B_{a_i} - F_{a_i}^*$  as before

$$X \xleftarrow{\eta} V \xrightarrow{i} Y = U_p \subset B_{a_i}$$

where  $\eta$  and  $i$  are strongly étale with  $i$  being surjective and where  $Y = U_p$  satisfies the condition (\*).

This gives rise to the commutative diagram of Cartesian products

$$\begin{array}{ccccccc}
X^{tor} & \xleftarrow{\eta^{tor}} & V^{tor} & \xrightarrow{i^{tor}} & Y^{tor} & \subset & B_{a_i}^{tor} \\
\mu_X \downarrow & & \mu_V \downarrow & & \mu_Y \downarrow & & \\
X & \xleftarrow{\eta} & V & \xrightarrow{i} & Y & \subset & B_{a_i}.
\end{array}$$

Since the canonical resolution is pulled-back by étale morphisms and since so does the canonical principalization of ideals, noting

$$\{\eta^{tor}\}^* \{\mu_X^{-1} I_X \cdot \mathcal{O}_{X^{tor}}\} = \{i^{tor}\}^* \{\mu_Y^{-1} I_Y \cdot \mathcal{O}_{Y^{tor}}\},$$

which implies

$$\begin{aligned}
\{\eta_-/K^*\}^* (I_{i-})_{X_-/K^*} &= \{i_-/K^*\}^* (I_{i-})_{Y_-/K^*} \\
\{\eta_+/K^*\}^* (I_{i+})_{X_+/K^*} &= \{i_+/K^*\}^* (I_{i+})_{Y_+/K^*},
\end{aligned}$$

we have the commutative diagram of Cartesian products

$$\begin{array}{ccccccc}
X_-/K^* & \leftarrow & V_-/K^* & \rightarrow & Y_-/K^* & \subset & W_i \\
(X_-/K^*)^{res} & \leftarrow & (V_-/K^*)^{res} & \rightarrow & (Y_-/K^*)^{res} & \subset & W_{i-}^{res} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(X^{tor})_-/K^* & \leftarrow & (V^{tor})_-/K^* & \rightarrow & (Y^{tor})_-/K^* & \subset & W_{i-}^{tor} \\
(X_-/K^*)^{can} & \leftarrow & (V_-/K^*)^{can} & \rightarrow & (Y_-/K^*)^{can} & \subset & W_{i-}^{can} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^{tor}/K^* & \leftarrow & V^{tor}/K^* & \rightarrow & Y^{tor}/K^* & \subset & B_{a_i}/K^* \\
(X^{tor})_+/K^* & \leftarrow & (V^{tor})_+/K^* & \rightarrow & (Y^{tor})_+/K^* & \subset & W_{i+}^{tor} \\
(X_+/K^*)^{can} & \leftarrow & (V_+/K^*)^{can} & \rightarrow & (Y_+/K^*)^{can} & \subset & W_{i+}^{can} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_+/K^* & \leftarrow & V_+/K^* & \rightarrow & Y_+/K^* & \subset & W_{i+} \\
(X_+/K^*)^{res} & \leftarrow & (V_+/K^*)^{res} & \rightarrow & (Y_+/K^*)^{res} & \subset & W_{i+}^{res}
\end{array}$$

Therefore, we only have to show the assertion assuming that the quasi-elementary cobordism  $B_{a_i} = X$  is a nonsingular affine toric variety.

Note first that the morphisms

$$\begin{aligned}
f_{i-} : W_{i-}^{tor} &= (X^{tor})_-/K^* \rightarrow W_i = X_-/K^* \\
f_{i+} : W_{i+}^{tor} &= (X^{tor})_+/K^* \rightarrow W_i = X_+/K^*
\end{aligned}$$

are equivariant (toric) ones between toric varieties by construction, that so are the morphisms

$$\begin{aligned}
r_{i-} : W_i^{res} &= (X_-/K^*)^{res} \rightarrow W_i = X_-/K^* \\
r_{i+} : W_i^{res} &= (X_+/K^*)^{res} \rightarrow W_{i+1} = X_+/K^*
\end{aligned}$$

since the canonical resolution of singularities preserves the action of the embedded torus (See the condition  $(\spadesuit^{res} - 1)$ ), that the morphisms

$$\begin{aligned}
p_{i-} : W_{i-}^{can} &= (X_-/K^*)^{can} \rightarrow W_i^{res} = (X_-/K^*)^{res} \\
p_{i+} : W_{i+}^{can} &= (X_+/K^*)^{can} \rightarrow W_i^{res} = (X_+/K^*)^{res}
\end{aligned}$$

since the ideals  $I_{i-}$  and  $I_{i+}$  (and their pull-backs by  $r_{i-}$  and  $r_{i+}$ ) are both toric by construction (cf. Lemma 4-1-2) and since the canonical principalization also preserves the action of the embedded torus (See the condition  $(\spadesuit^{can} - 1)$ ) and that hence so are the morphisms

$$\begin{aligned}
h_{i-} : W_{i-}^{can} &= (X_-/K^*)^{can} \rightarrow W_{i-}^{tor} = (X_-/K^*)^{tor} \\
h_{i+} : W_{i+}^{can} &= (X_+/K^*)^{can} \rightarrow W_{i+}^{tor} = (X_+/K^*)^{tor}.
\end{aligned}$$

We denote the corresponding fans and associated toric varieties by

$$\begin{aligned}
W_i &= X(\pi(N), \pi(\partial_-\sigma)) \\
W_i^{res} &= X(\pi(N), \Sigma_i^{res}) \\
W_{i-}^{tor} &= X(\pi(N), \Sigma_{i-}^{tor}) \\
W_{i-}^{can} &= X(\pi(N), \Sigma_{i-}^{can}) \\
W_{i+1} &= X(\pi(N), \pi(\partial_+\sigma)) \\
W_{i+1}^{res} &= X(\pi(N), \Sigma_{i+1}^{res}) \\
W_{i+}^{tor} &= X(\pi(N), \Sigma_{i+}^{tor}) \\
W_{i+}^{can} &= X(\pi(N), \Sigma_{i+}^{can})
\end{aligned}$$

Let  $\{\sigma_{Z_{max}}\}$  be the set of the maximal cones of  $\sigma^{tor}$ , where  $\sigma^{tor}$  is the fan corresponding to the toric variety  $X^{tor} = X(N, \sigma^{tor})$ . By Theorem 3-2-8 (i) we have the situation  $(\heartsuit_{fan})$ . Therefore, for any  $\sigma_{Z_{max}}$  we have the decomposition

$$\sigma_{Z_{max}} = \sum_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle v_j \rangle \oplus \tau_{Z'_{max}} \text{ as cones,}$$

where the cone  $\tau_{Z'_{max}}$  is generated by the extremal rays  $w'_k s$  (other than those  $v_j$ 's with  $j \notin S$ ) of  $\sigma_{Z_{max}}$  and is contained in a linear space  $L$  (of codimension  $\#\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}$ ) which also contains  $a \in N$  and where we have the exact sequence

$$0 \rightarrow L \cap N \rightarrow N \rightarrow \mathbb{Z}^{\oplus \#\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \rightarrow$$

with the image of  $\{v_j; j \notin S, v_j \in \sigma_{Z_{max}}\}$  forming a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^{\#\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}}$ .

Therefore, under the projection map  $\pi : N_{\mathbb{R}} \rightarrow N/a \cdot N_{\mathbb{R}}$  we have

$$\pi(N) = \sum_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \mathbb{Z} \cdot \pi(v_j) \oplus \pi(N \cap L)$$

and

$$\begin{aligned}
\pi(\partial_-\sigma_{Z_{max}}) &= \sum_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle \pi(v_j) \rangle \oplus \pi(\partial_- \tau_{Z'_{max}}) \\
\pi(\sigma_{Z_{max}}) &= \sum_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle \pi(v_j) \rangle \oplus \pi(\tau_{Z'_{max}}) \\
\pi(\partial_+\sigma_{Z_{max}}) &= \sum_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle \pi(v_j) \rangle \oplus \pi(\partial_+ \tau_{Z'_{max}}).
\end{aligned}$$

By Theorem 3-2-8 (iii) we have the commutative diagram of Cartesian products

$$\begin{array}{ccc}
X(\pi(N), \pi(\partial_-\sigma_{Z_{max}})) & & X(\pi(N), \pi(\partial_+\sigma_{Z_{max}})) \\
& \searrow & \swarrow \\
& \downarrow & \\
(X^{tor})_-/K^* & & (X^{tor})_+/K^* \\
& \nwarrow & \downarrow \\
& & X^{tor}/K^*.
\end{array}$$

Since the affine toric varieties  $\{X(\pi(N), \pi(\sigma_{Z_{max}}))\}$  cover  $X^{tor}/K^*$ , we conclude that  $\{X(\pi(N), \pi(\partial_-\sigma_{Z_{max}}))\}$  cover  $(X^{tor})_-/K^* = X(\pi(N), \Sigma_-^{tor})$ . That is to say, the cones  $\{\pi(\partial_-\sigma_{Z_{max}})\}$  cover the fan  $\Sigma_-^{tor}$ . Thus the fan  $\Sigma_-^{can}$ , which is a refinement of  $\Sigma_-^{tor}$ , is the union of the refinements  $\pi(\partial_-\sigma_{Z_{max}})^{can}$  of  $\pi(\partial_-\sigma_{Z_{max}})$ . Also we have the identical statement replacing the  $-$  sign with the  $+$  sign.

We prove the following situations  $(\heartsuit_{\pi(\partial_-fan)})$  and  $(\heartsuit_{\pi(\partial_+fan)})$  hold:

$$(\heartsuit_{\pi(\partial_-fan)}) \begin{cases} \pi(\partial_-\sigma_{Z_{max}})^{can} \text{ has the product structure} \\ \pi(\partial_-\sigma_{Z_{max}})^{can} = \sum_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle \pi(v_j) \rangle \oplus \pi(\partial_-\tau_{Z'_{max}})^{can} \\ \text{inherited from the product structure of } \pi(\partial_-\sigma_{Z_{max}}) \text{ and } \pi(\sigma_{Z_{max}}) \\ \pi(\partial_-\sigma_{Z_{max}}) = \sum_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle \pi(v_j) \rangle \oplus \pi(\partial_-\tau_{Z'_{max}}) \\ \pi(\sigma_{Z_{max}}) = \sum_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle \pi(v_j) \rangle \oplus \pi(\tau_{Z'_{max}}) \\ \text{where } \pi(\partial_-\tau_{Z'_{max}})^{can} \text{ is a refinement of } \pi(\partial_-\tau_{Z'_{max}}). \end{cases}$$

The situation  $(\heartsuit_{\pi(\partial_+fan)})$  is identical to  $(\heartsuit_{\pi(\partial_-fan)})$ , replacing the  $-$  sign with the  $+$  sign.

In fact, since the toroidal structure

$$X(\pi(N), \pi(\sigma_{Z_{max}})) \cap (U_{X^{tor}}, X^{tor}) = (U_{X(\pi(N), \pi(\sigma_{Z_{max}}))}, X(\pi(N), \pi(\sigma_{Z_{max}})))$$

is covered by the open sets of the form

$$\begin{aligned} & \prod_{\{j; j \in S, v_j \in \sigma_{Z_{max}}\}} (\mathbb{A}^1 - \{\delta_j\}, \mathbb{A}^1 - \{\delta_j\}) \times (T_{X(\pi(N \cap L), \pi(\partial_-\tau_{Z'_{max}}))}, X(\pi(N \cap L), \pi(\partial_-\tau_{Z'_{max}}))) \\ & \cong \prod_{\{j; j \in S, v_j \in \sigma_{Z_{max}}\}} (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (T_{X(\pi(N \cap L), \pi(\partial_-\tau_{Z'_{max}}))}, X(\pi(N \cap L), \pi(\partial_-\tau_{Z'_{max}}))), \end{aligned}$$

the situations  $(\heartsuit_{\pi(\partial_-fan)})$  and  $(\heartsuit_{\pi(\partial_+fan)})$  would imply that the morphisms  $h_{i-}$  and  $h_{i+}$  over these open sets are toric morphisms of the form

$$\begin{aligned} h_{i-} : & \prod (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (T_{X(\pi(N \cap L), \pi(\partial_-\tau_{Z'_{max}}))}, X(\pi(N \cap L), \pi(\partial_-\tau_{Z'_{max}})^{can})) \\ & \rightarrow \\ & \prod (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (T_{X(\pi(N \cap L), \pi(\partial_-\tau_{Z'_{max}}))}, X(\pi(N \cap L), \pi(\partial_-\tau_{Z'_{max}}))) \\ h_{i+} : & \prod (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (T_{X(\pi(N \cap L), \pi(\partial_+\tau_{Z'_{max}}))}, X(\pi(N \cap L), \pi(\partial_+\tau_{Z'_{max}})^{can})) \\ & \rightarrow \\ & \prod (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (T_{X(\pi(N \cap L), \pi(\partial_+\tau_{Z'_{max}}))}, X(\pi(N \cap L), \pi(\partial_+\tau_{Z'_{max}}))) \end{aligned}$$

where the morphisms on the first factors are identities and where the morphisms on the second factors are induced by the refinements of the cones and that hence  $h_{i-}$  and  $h_{i+}$  are toroidal. Since we have the diagram of toric morphisms

$$\begin{array}{ccc}
X(\pi(N \cap L), \pi(\partial_{-\tau_{Z'_{max}}})^{can}) & \xrightarrow{\varphi_i^{can}} & X(\pi(N \cap L), \pi(\partial_{+\tau_{Z'_{max}}})^{can}) \\
\downarrow & & \downarrow \\
X(\pi(N \cap L), \pi(\partial_{-\tau_{Z'_{max}}})) & & X(\pi(N \cap L), \pi(\partial_{+\tau_{Z'_{max}}})) \\
\searrow & & \swarrow \\
& X(\pi(N \cap L), \pi(\tau_{Z'_{max}})), &
\end{array}$$

we also conclude that  $\varphi_i^{can}$  is V-toroidal.

Now we only discuss the verification of the situation  $(\heartsuit_{\pi(\partial_- fan)})$ , as that of  $(\heartsuit_{\pi(\partial_+ fan)})$  is identical replacing the  $-$  sign with the  $+$  sign.

Note that, induced by the product structure

$$\sigma_{Z_{max}} = \langle v_j \rangle \oplus \tau_{Z'_{max,j}},$$

where the cone  $\tau_{Z'_{max,j}}$  is contained in a hyper plane  $H_j$  which also contains  $a \in N$  and where we have the exact sequence

$$0 \rightarrow H_j \cap N \rightarrow N \rightarrow \mathbb{Z} \rightarrow 0$$

with the image of  $v_j \in N$  to be  $\pm 1 \in \mathbb{Z}$ , we have the product structure

$$\pi(\partial_{-\sigma_{Z_{max}}}) = \langle \pi(v_j) \rangle \oplus \pi(\partial_{-\tau_{Z'_{max,j}}}).$$

Since

$$\pi(L) = \bigcap_{j \notin S, v_j \in \sigma_{Z_{max}}} \pi(H_j),$$

it is straightforward to observe that in order to prove the situation  $(\heartsuit_{\pi(\partial_- fan)})$  it suffices to show the collection of the situations  $(\heartsuit_{j, \pi(\partial_- fan)})$  for all  $j = 1, \dots, l$ :

$$(\heartsuit_{j, \pi(\partial_- fan)}) \left\{ \begin{array}{l} \text{Provided } j \notin S \text{ and } v_j \in \sigma_{Z_{max}}, \\ \pi(\partial_{-\sigma_{Z_{max}}})^{can} \text{ has the product structure} \\ \pi(\partial_{-\sigma_{Z_{max}}})^{can} = \langle \pi(v_j) \rangle \oplus \pi(\partial_{-\tau_{Z'_{max,j}}})^{can} \\ \text{inherited from the product structure of } \pi(\partial_{-\sigma_{Z_{max}}}) \text{ and } \pi(\sigma_{Z_{max}}) \\ \pi(\partial_{-\sigma_{Z_{max}}}) = \langle \pi(v_j) \rangle \oplus \pi(\partial_{-\tau_{Z'_{max,j}}}) \\ \pi(\sigma_{Z_{max}}) = \langle \pi(v_j) \rangle \oplus \pi(\tau_{Z'_{max,j}}) \\ \text{where } \pi(\partial_{-\tau_{Z'_{max,j}}})^{can} \text{ is a refinement of } \pi(\partial_{-\tau_{Z'_{max,j}}}). \end{array} \right.$$

Now we prove the situation  $(\heartsuit_{j, \pi(\partial_- fan)})$  ( $j = 1, \dots, l$ ) in the form of the following proposition.

**Proposition 4-1-4.** Let  $\pi(\partial_{-}\sigma_{Z_{max}})^{can}$  be the subfan of  $\Sigma_{-can}$  lying over the cone  $\pi(\partial_{-}\sigma_{Z_{max}})$  of  $\Sigma_{-}^{tor}$ . Let  $v_j$  ( $j = 1, \dots, l$ ) be such primitive vector with  $v_j \notin D^{tor}$  and  $v_j \in \sigma_{Z_{max}}$ . Then any new extremal ray  $w \in \pi(\partial_{-}\sigma_{Z_{max}})^{can}$  to subdivide  $\pi(\partial_{-}\sigma_{Z_{max}})$  sits inside of the cone  $\pi(\partial_{-}\tau_{Z'_{max,j}})$ , where  $\pi(\partial_{-}\sigma_{Z_{max}})$  has the product structure

$$\pi(\partial_{-}\sigma_{Z_{max}}) = \langle \pi(v_j) \rangle \oplus \pi(\partial_{-}\tau_{Z'_{max,j}})$$

induced from the product structure of  $\sigma_{Z_{max}}$

$$\sigma_{Z_{max}} = \langle v_j \rangle \oplus \tau_{Z'_{max,j}}.$$

Therefore, we have the situation  $(\heartsuit_{j,\pi(\partial_{-}fan)})$ .

proof.

We use the same notation as in Proposition 3-1-8 or Observation 3-2-1.

We take a  $\mathbb{Z}$ -basis  $\{v_1, \dots, v_n\}$  of the lattice  $N$  so that

$$\begin{aligned} \sigma &= \langle v_1, \dots, v_l \rangle \\ \sigma^{\vee} &= \langle v_1^*, \dots, v_l^*, \pm v_{l+1}^*, \dots, \pm v_n^* \rangle. \end{aligned}$$

Setting  $z_i = z^{v_i^*}$ , we have

$$X = \text{Spec } K[z_1, \dots, z_l, z_{l+1}^{\pm 1}, \dots, z_n^{\pm 1}].$$

The  $K^*$ -action corresponds to a one-parameter subgroup

$$a = (\alpha_1, \dots, \alpha_n) \in N,$$

i.e., the action of  $t \in K^*$  is given by

$$t \cdot (z_1, \dots, z_n) = (t^{\alpha_1} z_1, \dots, t^{\alpha_n} z_n).$$

Then by the proof of Theorem 3-2-8 the affine toric variety associated to a maximal cone  $\sigma_{Z_{max}} \in \sigma^{tor}$  has the form

$$\text{Spec}\{K[v_j^* - m_1] \otimes K[\{y_h\}]\}$$

where the monomials  $\{y_h\}$  are taken from  $v_j^{\perp} \cap M$  and where in the exact sequence

$$0 \rightarrow v_j^{\perp} \cap M \rightarrow M \rightarrow \mathbb{Z} \rightarrow 0$$

the image of  $v_j^* - m_1 \in M$  is  $\pm 1 \in \mathbb{Z}$ .

As the dual statement in terms of the cones, we obtain the product structure

$$\sigma_{Z_{max}} = \langle v_j \rangle \oplus \tau_{Z'_{max,j}}$$

where the cone  $\tau_{Z'_{max,j}}$  is defined by the equation  $(v_j^* - m_1, \cdot) = 0$ . Therefore, in the induced product structure

$$\Sigma_{-}^{tor} \supset \partial_{-}\sigma_{Z_{max}} = \langle \pi(v_j) \rangle \oplus \partial_{-}\tau_{Z'_{max,j}}$$

the cone  $\partial_- \tau_{Z'_{max,j}}$  is also defined by the equation  $(v_j^* - m_1, \cdot) = 0$ . (Note that since  $(v_j^* - m_1, a) = 0$ , we have  $v_j^* - m_1 \in M \cap a^\perp = \text{Hom}_{\mathbb{Z}}(\pi(N), \mathbb{Z})$ .)

Therefore, in order to show that a new extremal ray  $w \in \pi(\partial_- \sigma_{Z_{max}})^{can}$  to subdivide  $\pi(\partial_- \sigma_{Z_{max}})$  sits inside of  $\partial_- \tau_{Z'_{max,j}}$  it suffices to show

$$(v_j^* - m_1, w) = \text{ord}_{E_w}(z_j \cdot z^{-m_1}) \leq 0,$$

where  $\text{ord}_{E_w}(r)$  denotes the order of the rational function  $r$  on the divisor (necessarily exceptional for  $h_{i-}$ ) corresponding to the ray  $w$ .

Now consider the family of automorphisms  $\theta_c$  parametrized by  $c \in K$  of the ring  $K[z_1, \dots, z_l, z_{l+1}^{\pm 1}, \dots, z_n^{\pm 1}]$  (or equivalently the action of  $K = \mathbb{A}^1$  on  $\text{Spec}K[z_1, \dots, z_l, z_{l+1}^{\pm 1}, \dots, z_n^{\pm 1}]$ ) defined by

$$\begin{aligned}\theta_c^*(z_j) &= z_j + c \cdot z^{m_1} \\ \theta_c^*(z_k) &= z_k \text{ for } k \neq j.\end{aligned}$$

Observe that these automorphisms preserve the toric ideals because  $z_j$  and  $m_1$  has the same character. It follows that  $\theta_c$  induces a family of automorphisms of  $W_{i-}^{tor}$  mapping the divisor  $E_{i-}^{tor}$  to itself. Therefore, the automorphisms also preserve  $I_{i-}$ . (See Lemma 4-1-2.)

Therefore, by the properties  $(\spadesuit^{res} - 1)$  and  $(\spadesuit^{can} - 1)$  of the canonical resolution and canonical principalization, we conclude that  $\theta_c$  induces an automorphism of  $W_{i-}^{can}$  which preserves the exceptional divisor  $E_w$ . (Note that the set of exceptional divisors for  $h_{i-}$  are discrete, while the family is parametrized by a continuous family  $\mathbb{A}^1$ . See Remark 4-1-1.)

Then, denoting the induced automorphism by the same  $\theta_c$  by abuse of notation, we have

$$\begin{aligned}\text{ord}_{E_w}(z_1 \cdot z^{-m_1}) &= \text{ord}_{E_w}(\theta_c^*(z_1 \cdot z^{-m_1})) \\ &= \text{ord}_{E_w}((z_1 + c \cdot z^{m_1}) \cdot z^{-m_1}) \\ &= \text{ord}_{E_w}(z_1 \cdot z^{-m_1} + c).\end{aligned}$$

Since the above equality holds for any  $c \in K$ , we conclude

$$\text{ord}_{E_w}(z_1 \cdot z^{-m_1}) \leq 0 \text{ (and hence } = 0)$$

as required.

This completes the proof of Proposition 4-1-4.

#### §4-2. Conclusion of the proof for the Weak Factorization Theorem

Since  $X_1 = W_1$  and  $W_s = X_2$  are nonsingular, we have

$$X_1 = W_1 \xleftarrow{p_{1-}} W_1^{res} \text{ and } W_l^{res} \xrightarrow{p_{s-1,-}} W_s = X_2.$$

For each  $i = 1, \dots, s-1$  we have constructed the commutative diagram

$$\begin{array}{ccccccc} W_i^{res} & \xleftarrow{p_{i-}} & W_{i-}^{can} & \xrightarrow[\dashrightarrow]{\varphi_i^{can}} & W_{i+}^{can} & \xrightarrow{p_{i+}} & W_{i+1}^{res} \\ r_{i-} \downarrow & & h_{i-} \downarrow & & \downarrow h_{i+} & & \downarrow r_{i+} \\ W_i & \xleftarrow{f_{i-}} & W_{i-}^{tor} & \xrightarrow[\dashrightarrow]{\varphi_i^{tor}} & W_{i+}^{tor} & \xrightarrow{f_{i+}} & W_{i+1} \end{array}$$

where

- (i)  $W_i^{res}, W_{i-}^{can}, W_{i+}^{can}$  and  $W_{i+1}^{res}$  are all nonsingular,
- (ii) the morphisms  $p_{i-}$  and  $p_{i+}$  are sequences of blowups with smooth centers, and
- (iii)  $W_{i-}^{can}$  and  $W_{i+}^{can}$  have toroidal structures so that

$$\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$$

is V-toroidal.

**Theorem 4-2-1 (Strong Factorization Theorem for Toroidal Birational Maps).** *Let*

$$\varphi : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$$

*be a proper and toroidal birational map between nonsingular toroidal embeddings. Then  $\varphi$  can be factored into a sequence of toroidal blowups immediately followed by toroidal blowdowns with smooth centers.*

*Proof.*

If  $\varphi : (U_{W_1}, W_1) \rightarrow (U_{W_2}, W_2)$  is a proper and toroidal birational morphism between nonsingular toroidal embeddings without self-intersections (which is necessarily allowable in the sense of KKMS [1] (cf. Abramovich-Karu [1] or Chapter 2)), then the theorem is exactly what is proved in Abramovich-Matsuki-Rashid [1].

In general, by definition of a proper birational map  $\varphi$  being toroidal there exists a toroidal embedding  $(U_V, V)$  which dominates  $(U_{W_1}, W_1)$  and  $(U_{W_2}, W_2)$  by proper and toroidal birational morphisms

$$(U_{W_1}, W_1) \xleftarrow{\varphi_1} (U_V, V) \xrightarrow{\varphi_2} (U_{W_2}, W_2).$$

(Note that in case  $\varphi$  is V-toroidal we can take  $(U_V, V)$  to consist of the normalization  $V$  of the graph  $\Gamma_\varphi \subset W_1 \times W_2$  and the open set  $U_V = p_1^{-1}(U_{W_1}) = p_2^{-1}(U_{W_2})$  where  $p_1 : V \rightarrow W_1$  and  $p_2 : V \rightarrow W_2$  are the projections.)

By Lemma 1-2-6 there exists a sequence of toroidal blowups of  $(U_{W_1}, W_1)$  with smooth centers so that the result is a toroidal embedding without self-intersection. Since all the blowups are toroidal and since  $\varphi_1$  is also toroidal, we can take the corresponding sequence of toroidal blowups of  $(U_V, V)$  so that the result dominates the blowup of  $(U_{W_1}, W_1)$ . We can apply the same procedure to  $\varphi_2$ . By eliminating the self-intersection of and resolving the singularities of  $(U_V, V)$  by a sequence of toroidal blowups, we finally conclude that we may assume that  $(U_{W_1}, W_1), (U_V, V)$  and  $(U_{W_2}, W_2)$  are all nonsingular toroidal embeddings without self-intersections. (Note that in case  $\varphi$  is V-toroidal, then  $\varphi_1$  and  $\varphi_2$  are automatically allowable by construction, without referring to the result of Abramovich-Karu [1] (cf. Lemma 1-2-7).)

Now we are in the situation described as above to apply the strong factorization theorem to factor  $\varphi_1$  as  $(U_{W_1}, W_1) \xleftarrow{\mu_1} (U_{\tilde{V}}, \tilde{V}) \xrightarrow{\mu_V} (U_V, V)$  where  $\mu_1$  and  $\mu_V$  are toroidal blowups with smooth centers. Now we only have to apply the strong factorization theorem to  $\varphi_2 \circ \mu_V$  to complete the proof for Theorem 4-2-1.

We apply Theorem 4-2-1 to the proper and V-toroidal birational map  $\varphi_i^{can}$  to finish the proof of the weak factorization theorem.

**(Preservation of the open set where  $\phi$  is an isomorphism)**

Suppose  $\phi : X_1 \dashrightarrow X_2$  induces an isomorphism on an open set  $\phi : X_1 \supset U \xrightarrow{\sim} U \subset X_2$ . Then

Step 1: In the process of eliminating the points of indeterminacy to be reduced to the case where  $\phi$  is a projective birational morphism, all the centers of blowups are taken outside of  $U$ .

Step 2: In the process of constructing the cobordism  $B_\phi(X_1, X_2)$  out of the product  $X_2 \times \mathbb{P}^1$ , all the centers of blowups are taken outside of  $U \times \mathbb{P}^1$ .

Step 3: In the process of torification, since the torific ideals are trivial over  $U \times \mathbb{P}^1 \cap B_{a_i}$  for all  $i$ , all the modifications are made outside of  $U \times \mathbb{P}^1 \cap B_{a_i}$ .

Therefore, through Steps 1,2 and 3, the open set  $U$  remains untouched inside of all the  $W_i$  and  $U_{W_{i\pm}^{tor}} \subset W_{i\pm}^{tor}$ .

Step 4: In the process of obtaining the  $W_i^{res}$  or  $W_{i+1}^{res}$  from  $W_{i\pm}^{tor}$  all the centers of blowups are taken over the singular locus and hence outside of  $U$ .

In the process of obtaining the  $W_{i\pm}^{can}$  from  $W_i^{res}$  or  $W_{i+1}^{res}$  all the centers are taken over the quotient (by the  $K^*$ -action) of the locus where the torific ideals are not trivial and hence outside of  $U$ .

Step 5: Finally in the process of factoring the toroidal birational maps  $\varphi_i^{can} : (U_{W_i^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$  all the centers of blowups are taken within the boundary divisors and hence outside of  $U$ .

Thus we conclude that all through the construction of our factorization the open set  $U$  is preserved without being modified.

**(Preservation of the projectivity)**

Step 1: In the process of eliminating the points of indeterminacy to be reduced to the case where  $\phi$  is a projective morphism via Lemma 1-4-2, by replacing  $\phi : X_1 \rightarrow X_2$  with  $\phi' : X'_1 \rightarrow X'_2$ ,  $X'_1$  as well as all the intermediate varieties in the sequence of blowups starting from  $X_1$  are projective over  $X_1$ , while  $X'_2$  as well as all the intermediate varieties in the sequence of blowups starting from  $X_2$  are projective over  $X_2$ .

Step 2: In the process of constructing the birational cobordism  $B_{\phi'}(X'_1, X'_2)$ , all the  $W_i$  are projective over  $X'_2$  via the Geometric Invariant Theory interpretation in Chapter 2.

Steps 3, 4: All the  $W_{i\pm}^{tor}, W_i^{res}$  and  $W_{i\pm}^{can}$  are also projective over  $X'_2$ , as they are obtained from  $W_i$  by sequences of blowing ups.

Step 4: Now by the use of the STRONG factorization theorem applied to the toroidal birational maps  $\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$  all the varieties between  $W_{i-}^{can}$  and  $W_{i+}^{can}$  are also projective over  $X'_2$  in the factorization.

Thus we conclude that in the factorization we construct there is an index  $i_o$  such that all the intermediate varieties  $V_i$  for  $i \leq i_o$  are projective over  $X_1$  and that  $V_i$  for  $i_o \leq i$  are projective over  $X_2$ . In particular, if both  $X_1$  and  $X_2$  are projective, they are all projective.

This completes the proof of the Weak Factorization Theorem and Chapter 4.

## CHAPTER 5. GENERALIZATIONS

In this chapter we provide several generalizations of the weak factorization theorem, specifying the modifications we have to make to the arguments in Steps 1 through 5 of the strategy of the proof (cf. §0-1. Outline of the strategy) according to the different purposes.

In §5-1, we establish the weak factorization theorem for bimeromorphic maps between compact complex manifolds. Except for a couple of places where we have to avoid the use of theorems only applicable in the algebraic category and replace them with some arguments valid in the analytic category, the proof goes with little change and in fact is even simpler. For example, instead of discussing with Luna's locally toric charts where we have to require the property of being strongly étale in the algebraic category, we can take  $\mathbb{C}^*$ -equivariant analytic isomorphisms as local charts where the induced analytic isomorphisms for the quotients are automatic.

In §5-2, we consider the case where a birational map  $\phi : X_1 \dashrightarrow X_2$  is equivariant with respect to the actions of a group on  $X_1$  and  $X_2$  and establish the equivariant weak factorization theorem. (Remark that the action does NOT have to be over the base field  $K$ .) We note here that in order to preserve the equivariance we may have to blow up several smooth irreducible centers simultaneously in the equivariant factorization. Since most of the constructions in our proof are canonical, the equivariance comes almost free under the group action in the main steps of the strategy and the proof is reduced to the equivariant version of the factorization of toroidal birational maps, for which we give a proof utilizing an idea of Abramovich-Wang [1] and hence generalizing the result of Morelli [1][2] Abramovich-Matsuki-Rashid [1].

In §5-3, we establish the weak factorization theorem over a base field  $K$  which may NOT be algebraically closed. By taking the algebraic closure  $\overline{K}$  of  $K$  and considering the action of the Galois group  $G = \text{Gal}(\overline{K}/K)$  after base change, it is almost a direct corollary of the arguments for the equivariant case, except for the technical difficulty of dealing with the possibility that after base change  $X_1 \times_{\text{Spec } K} \text{Spec } \overline{K}$  and  $X_2 \times_{\text{Spec } K} \text{Spec } \overline{K}$  may not stay irreducible but split into several smooth irreducible components.

In §5-4, we establish the weak factorization theorem in the logarithmic category, factoring a proper birational map between nonsingular toroidal embeddings into blowups and blowdowns with smooth and ADMISSIBLE centers. This has an application to the study of the behavior of the Hodge structures under birational transformations, e.g., it provides an easy proof for a theorem of Batyrev [1] claiming the invariance of the Betti numbers among birational nonsingular minimal models.

In §5-5, we discuss the toroidalization and strong factorization conjectures in the vicinity of the circle of ideas involving the factorization problem, resolution of singularities, semi-stable reductions and the log category of Kato [1].

### §5-1. Factorization of bimeromorphic maps

**Theorem 5-1-1 (Weak Factorization Theorem of Bimeromorphic Maps).** *Let  $\phi : X_1 \dashrightarrow X_2$  be a bimeromorphic map between compact complex manifolds. Let  $X_1 \supset U \subset X_2$  be a common open subset over which  $\phi$  is an isomorphism. Then  $\phi$*

can be factored into a sequence of blowups and blowdowns with smooth irreducible centers disjoint from  $U$ . That is to say, there exists a sequence of bimeromorphic maps between compact complex manifolds

$$X_1 = V_1 \xrightarrow{\psi_1} V_2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{i-1}} V_i \xrightarrow{\psi_i} V_{i+1} \xrightarrow{\psi_{i+1}} \cdots \xrightarrow{\psi_{l-2}} V_{l-1} \xrightarrow{\psi_{l-1}} V_l = X_2$$

where

- (i)  $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$ ,
- (ii)  $\psi_i$  are isomorphisms over  $U$ , and
- (iii) either  $\psi_i : V_i \dashrightarrow V_{i+1}$  or  $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$  is a morphism obtained by blowing up a smooth irreducible center disjoint from  $U$ .

Moreover, if both  $X_1$  and  $X_2$  are projective, then we can choose a factorization so that all the intermediate complex manifolds  $V_i$  are projective.

*Proof.*

We only specify the modifications we have to make at each step of the strategy for the proof described in Chapter 0. Introduction. (Note that if  $X_1$  and  $X_2$  are projective, then both are algebraic and hence so is  $\phi$ . Thus the “Moreover” part is already proved but is stated explicitly again only for the sake of completeness.)

Step 1. Elimination of points of indeterminacy

Thanks to Hironaka [3], Lemma 1-4-1 holds without any change in the category of compact complex manifolds. Note that by construction there exists a line bundle of the form  $\mathcal{A} = \mathcal{O}_{X'_1}(\Sigma - a_i E_i)$ , where the  $E_i$  are  $g_1$ -exceptional divisors with  $a_i > 0$ , such that  $\mathcal{A}$  is relatively ample for  $\phi' : X'_1 \rightarrow X'_2$ , i.e., we have a projective embedding over  $X'_2$

$$\phi' : X'_1 \hookrightarrow \mathbb{P}(\phi'_*(\mathcal{A}^{\otimes l})) \rightarrow X'_2 \text{ for some sufficiently large } l \in \mathbb{N}.$$

Note that  $J = \phi'_*(\mathcal{A}^{\otimes l}) = \phi'_*\mathcal{O}_{X'_1}(l \cdot (\Sigma - a_i E_i)) \subset \phi'_*\mathcal{O}_{X'_1} = \mathcal{O}_{X'_2}$  is an ideal sheaf of  $\mathcal{O}_{X'_2}$ , which coincides with  $\mathcal{O}_{X'_2}$  outside of  $U$ .

Therefore, by replacing  $\phi : X_1 \dashrightarrow X_2$  with  $\phi' : X'_1 \rightarrow X'_2$ , we may assume as in the algebraic case that  $\phi : X_1 \rightarrow X_2$  is a projective morphism which is the blowup of  $X_2$  along an ideal sheaf  $J \subset \mathcal{O}_{X_2}$  with the support of  $\mathcal{O}_{X_2}/J$  being disjoint from  $U$ .

Step 2. Construction of a birational cobordism

Theorem 2-2-2 holds for a projective bimeromorphic morphism  $\phi : X_1 \rightarrow X_2$  described as in Step 1. Though the proof goes almost without any change, we show the existence of such a coherent sheaf  $\mathcal{E}$  as described in the assertion (ii) with the splitting  $\mathcal{E} = \bigoplus_{b \in \mathbb{Z}} \mathcal{E}_b$  into the eigenspaces inductively as follows (since we would like to avoid the complete reducibility argument for a coherent sheaf  $\mathcal{E}$  under a  $K^*$ -action, which is directly applicable only to an algebraic coherent sheaf under an algebraic  $K^*$ -action):

Recall that we obtain  $\overline{B} = W_l$  by a sequence of successive blowups with  $\mathbb{C}^*$ -invariant centers:

$$\overline{B} = W_l \rightarrow \cdots \rightarrow W_{i+1} \xrightarrow{\mu_i} W_i \rightarrow \cdots \rightarrow W_1 = W \rightarrow W_0 = X_2 \times \mathbb{P}^1.$$

We denote the induced projection onto  $X_2$  by

$$\tau_i : W_i \rightarrow X_2.$$

We start with the product  $W_0 = X_2 \times \mathbb{P}^1$ . Here we obviously have a relatively (very) ample line bundle  $\mathcal{L}_0 = p_2^* \mathcal{O}_{\mathbb{P}^1}$  and a coherent sheaf  $\mathcal{E}_0 = (\tau_0)_* \mathcal{L}_0$  with the splitting

$$\begin{aligned} \mathcal{E} &= (\tau_0)_* \mathcal{L}_0 = (p_1)_* \mathcal{L}_0 \\ &= \bigoplus_{b \in \mathbb{Z}} (\mathcal{E}_0)_b = p_1^* \mathcal{O}_{X_2} \cdot T_0 \oplus p_1^* \mathcal{O}_{X_2} \cdot T_1 \end{aligned}$$

where  $T_0$  and  $T_1$  are the homogeneous coordinates of  $\mathbb{P}^1$ , on which  $t \in K^*$  acts as  $t^*(T_0) = T_0$  and  $t^*(T_1) = t \cdot T_1$  by definition.

Suppose  $W_i$  is embedded equivariantly into  $\mathbb{P}(\mathcal{E}_i)$  where  $\mathcal{E}_i = (\tau_i)_* \mathcal{L}_i$  is a coherent sheaf with the splitting  $\mathcal{E}_i = \bigoplus_{b \in \mathbb{N}} (\mathcal{E}_i)_b$  into the eigenspaces for some relatively (very) ample line bundle  $\mathcal{L}_i$  with a  $\mathbb{C}^*$ -action.

Since the center of the blowup  $W_{i+1} \xrightarrow{\mu_i} W_i$  is  $\mathbb{C}^*$ -invariant (See the property ( $\spadesuit^{res} - 1$ ) of the canonical resolution.), we conclude the following: The defining ideal sheaf  $I_i$  of the center has the property that there exists a finite open covering  $\{U\}$  of  $X_2$  and a sufficiently large integer  $k_i \in \mathbb{N}$  such that

$$(\tau_i)^* (\tau_i)_* \{I_i \otimes \mathcal{L}_i^{\otimes k_i}\} \rightarrow I_i \otimes \mathcal{L}_i^{\otimes k_i}$$

is surjective and that over each open set  $U$  of the covering the sheaf  $(\tau_i)^* (\tau_i)_* \{I_i \otimes \mathcal{L}_i^{\otimes k_i}\}$  is generated by a finite number of sum of products, each of which has the same character as the others, of elements in the eigenspaces  $\Gamma(U, (\mathcal{E}_i)_b)$  with coefficients in  $\Gamma(U, \mathcal{O}_{X_2})$ . It follows immediately that if we set

$$\begin{aligned} \mathcal{L}_{i+1} &= \mu_i^* I_i \cdot \mathcal{O}_{W_{i+1}} \otimes \mu_i^* \mathcal{L}_i^{\otimes k_i} \\ \mathcal{E}_{i+1} &= (\tau_{i+1})_* \mathcal{L}_{i+1} \end{aligned}$$

then  $\mathcal{L}_{i+1}$  has a natural  $K^*$ -action with the splitting  $\mathcal{E}_{i+1} = \bigoplus_{b \in \mathbb{N}} (\mathcal{E}_{i+1})_b$  and the equivariant embedding  $W_{i+1} \hookrightarrow \mathbb{P}(\mathcal{E}_{i+1})$ .

The rest of the proof for Theorem 2-2-2 is identical to the one in the algebraic setting.

The GIT interpretation of the cobordism  $B$ , obtained from the compactified birational (bimeromorphic) cobordism  $\overline{B}$  projective over  $X_2$  constructed as above, holds without any change in the analytic setting, leading to the definition of the quasi-elementary pieces  $B_{a_i}$ .

In order to reduce the analysis in general to the one on toric varieties, we take Luna's locally toric (toroidal) charts in the algebraic setting. In order to show the existence of these charts, we used some theorems of Sumihiro (Equivariant completion theorem & Covering by  $K^*$ -invariant affine spaces), Luna's Fundamental Lemma and (or) Luna's Étale Slice Theorem, all of which are directly applicable only in the algebraic setting (cf. Proposition 1-3-4 and Lemma 2-4-4). We replace Luna's locally toric charts by  $\mathbb{C}^*$ -equivariant analytic isomorphisms via the following lemma.

**Lemma 5-1-2.** *Let  $B_{a_i}$  be a quasi-elementary cobordism associated to the birational (bimeromorphic) cobordism  $B$  obtained from the compactified birational (bimeromorphic) cobordism  $\overline{B}$  projective over  $X_2$  constructed as above. Then for every point  $p \in B_{a_i} - F_{a_i}^*$ , where  $F_{a_i} = B_{a_i}^{\mathbb{C}^*}$  is the fixed point set and  $F_{a_i}^*$  is the set as defined in Notation 2-1-3, there exists a  $\mathbb{C}^*$ -equivariant analytic isomorphism*

$$X_p \supset V_p \xleftarrow{\eta_p} U_p \subset B_{a_i}$$

where  $X_p$  is a nonsingular affine toric variety with  $\mathbb{C}^*$ -acting as a one-parameter subgroup and where  $U_p \subset B_{a_i}$  (resp.  $V_p \subset X_p$ ) is a  $\mathbb{C}^*$ -invariant open subset (with respect to the usual topology) satisfying the following condition ( $\star$ ):

( $\star$ ) If a  $\mathbb{C}^*$ -orbit  $O(q)$  lies in  $U_p$  (resp.  $V_p$ ), then its closure  $\overline{O(q)}$  in  $B_{a_i}$  (resp. in  $X_p$ ) also lies in  $U_p$  (resp. in  $V_p$ ).

(As a consequence, we obtain

$$X_p//\mathbb{C}^* \supset V_p//\mathbb{C}^* \xleftarrow{\sim} U_p//\mathbb{C}^* \subset B_{a_i}//K^*$$

providing not only  $\mathbb{C}^*$ -equivariant analytic local isomorphisms of  $B_{a_i}$  to (nonsingular) toric varieties but also analytic local isomorphisms of the quotient  $B_{a_i}//\mathbb{C}^*$  to (not necessarily nonsingular) toric varieties.)

*Proof.*

It follows from Theorem 2-2-2 that, locally over  $X_2$ ,  $\overline{B}$  can be embedded equivariantly into  $X_2 \times \mathbb{P}^N$ , i.e., there exists a open neighborhood  $U_{X_2}$  of  $\tau_l(p)$  such that

$$\tau_l^{-1}(U_{X_2}) \hookrightarrow U_{X_2} \times \mathbb{P}^N,$$

where  $\mathbb{C}^*$  acts only on the second factor  $\mathbb{P}^N$  with  $t \in \mathbb{C}^*$  acting on the homogeneous coordinates  $T_0, \dots, T_N$  of  $\mathbb{P}^N$  by characters

$$t^*(T_j) = t^{b_j} \cdot T_j.$$

Case:  $p \in B_{a_i} - F_{a_i}^*$  is a fixed point, i.e.,  $p \in F_{a_i}$ .

In this case,  $T_j(p) = 0$  if  $b_j \neq a_i$ . Take  $j_o$  with  $T_{j_o}(p) \neq 0$  and  $b_{j_o} = a_i$ . Since  $p \in B_{a_i} \cap \tau_l^{-1}(U_{X_2}) \subset U_{X_2} \times \mathbb{P}^N$  is a nonsingular point, we may choose a regular coordinate system at  $p$  consisting of a regular coordinate system at  $\tau_l(p)$  of  $U_{X_2}$  and  $\{T_j/T_{j_o}; j \in J\}$  for some subset  $J \subset \{0, 1, \dots, N\}$ . Consider the  $\mathbb{C}^*$ -equivariant morphism

$$\eta : B_{a_i} \cap \tau_l^{-1}(U_{X_2}) \cap \{T_{j_o} \neq 0\} (\hookrightarrow U_{X_2} \times \mathbb{P}^N \cap \{T_{j_o} \neq 0\}) \rightarrow \mathbb{A}^{\dim X_2 + \#J} = X_p$$

where  $\mathbb{A}^{\dim X_2 + \#J}$  is the nonsingular affine toric variety  $X_p$  with affine coordinates corresponding to the regular coordinate system at  $p$  chosen as above and the action of  $t \in \mathbb{C}^*$  is given by

$$t^*(T_j/T_{j_o}) = t^{b_j - b_{j_o}} \cdot T_j/T_{j_o},$$

leaving others  $\mathbb{C}^*$ -invariant. By construction, there exist open neighborhoods

$$\begin{aligned} p &\in U_p''' \subset B_{a_i} \cap \tau_l^{-1}(U_{X_2}) \cap \{T_{j_o} \neq 0\} \\ \eta(p) &\in V_p''' \subset X_p \end{aligned}$$

such that  $\eta$  induces an analytic isomorphism between them

$$\eta|_{U_p'''} : U_p''' \xrightarrow{\sim} V_p'''.$$

Let  $\mathbb{C}_1^* = \{z \in \mathbb{C}^*; |z| = 1\}$  be the unit circle in  $\mathbb{C}^*$ . Since  $p$  is a fixed point for the action of  $\mathbb{C}^*$ , we may choose a small open neighborhood (with respect to the usual topology)  $p \in U_p'' \subset U_p'''$  such that  $t(q) \in U_p'''$  for all  $q \in U_p''$  and  $t \in \mathbb{C}_1^*$ . This implies that

$$Stab(q) = Stab(\eta(q)) \text{ for all } q \in U_p''.$$

Let

$$\begin{aligned} U'_p &= \cup_{t \in \mathbb{C}^*} t(U_p'') \\ V'_p &= \cup_{t \in \mathbb{C}^*} t(\eta(U_p'')) \end{aligned}$$

be the two  $\mathbb{C}^*$ -invariant neighborhood of  $p$  and  $\eta(p)$ , respectively. From the construction it follows that

$$\eta|_{U'_p} : U'_p \xrightarrow{\sim} V'_p$$

is a  $\mathbb{C}^*$ -equivariant analytic isomorphism.

Let  $J' = \{j \in J; b_j - b_{j_o} \neq 0\}$  be the subset of  $J$ . Then  $F_{X_p}$  has the description

$$F_{X_p} = \{x \in X_p; T_j/T_{j_o}(x) = 0 \text{ for } j \in J'\}.$$

Let  $\sigma : X_p = \mathbb{A}^{\dim X_2 + \#J} \rightarrow \mathbb{A}^{\dim X_2 + \#J'}$  be the obvious projection given by  $\prod_{j \in J} T_j/T_{j_o}(x) \mapsto \prod_{j \in J'} T_j/T_{j_o}(x)$ , leaving the rest of the coordinates coming from  $U_{X_2}$  unchanged. Take a small open neighborhood  $W \subset \mathbb{A}^{\dim X_2 + \#J'}$  of  $\sigma \circ \eta(p)$  so that  $\sigma^{-1}(W) \cap F_{X_p} \subset V'_p$ . It follows that

$$\begin{aligned} U_p &= U'_p \cap \eta^{-1}(V'_p \cap \sigma^{-1}(W)) \subset B_{a_i} \\ V_p &= V'_p \cap \sigma^{-1}(W) \subset X_p \end{aligned}$$

are  $\mathbb{C}^*$ -invariant open subsets satisfying the condition  $(\star)$  with a  $\mathbb{C}^*$ -equivariant analytic isomorphism

$$\eta_p : U_p \rightarrow V_p.$$

Case:  $p \in B_{a_i} - F_{B_{a_i}}^*$  is NOT a fixed point, i.e.,  $p \notin F_{a_i}$ .

Take a limit point  $q = \lim_{t \rightarrow 0} t(p)$  or  $\lim_{t \rightarrow \infty} t(p) \in \overline{B}$  so that  $q \in F_{\overline{B}}$  is a fixed point of  $\overline{B}$ . By the same argument as in the previous case, we can find a  $\mathbb{C}^*$ -equivariant analytic isomorphism

$$\eta_q : U'_q \rightarrow V'_q$$

where  $U'_q \subset \overline{B}$  and  $V'_q \subset X_q$  are  $\mathbb{C}^*$ -invariant open neighborhoods of  $q \in \overline{B}$  and  $\eta_q(q) \in X_q$ , respectively, and where  $X_q$  is a nonsingular affine toric variety with  $\mathbb{C}^*$  acting as a one-parameter subgroup.

Let  $p_2 : U_{X_2} \times \mathbb{P}^N \rightarrow \mathbb{P}^N$  be the second projection. Since  $p \in B_{a_i} - F_{B_{a_i}}^*$  and  $p \notin F_{B_{a_i}}$ , we conclude  $p_2(p) \in (\mathbb{P}^N)_{a_i} - F_{(\mathbb{P}^N)_{a_i}}^*$  and  $p_2(p) \notin F_{(\mathbb{P}^N)_{a_i}}$ , noting that

$$\begin{aligned}\tau_l^{-1}(U_{X_2}) \cap B_{a_i} &= \tau_l^{-1}(U_{X_2}) \cap \overline{B} \cap p_2^{-1}((\mathbb{P}^N)_{a_i}) \text{ and} \\ \tau_l^{-1}(U_{X_2}) \cap F_{B_{a_i}} &= \tau_l^{-1}(U_{X_2}) \cap B_{a_i} \cap p_2^{-1}(F_{(\mathbb{P}^N)_{a_i}}).\end{aligned}$$

In the proof of Proposition 1-3-5, we already saw that there exists an affine  $\mathbb{C}^*$ -invariant open neighborhood  $p \in U_{(\mathbb{P}^N)_{a_i}} \subset (\mathbb{P}^N)_{a_i}$  such that  $U_{(\mathbb{P}^N)_{a_i}}$  satisfies the condition  $(\star)$  and that  $U_{(\mathbb{P}^N)_{a_i}}$  has no fixed points.

Note also that since  $\eta_q(p)$  is not a fixed point in  $X_q$ , there exists a nonsingular affine toric open subvariety  $X_p \subset X_q$  such that  $\eta_q(p) \in X_p$ .

It follows that

$$\begin{aligned}U_p &= U'_p \cap p_2^{-1}(U_{(\mathbb{P}^N)_{a_i}}) \cap \eta_q^{-1}(X_p) \subset B_{a_i} \\ V_p &= \eta_q(U_p) \subset X_p\end{aligned}$$

are  $\mathbb{C}^*$ -invariant open subsets satisfying the condition  $(\star)$  with a  $\mathbb{C}^*$ -equivariant analytic isomorphism  $\eta_p = \eta_q|_{U_p} : U_p \xrightarrow{\sim} V_p$ .

This completes the proof of Lemma 5-1-2.

It is worthwhile noting that, in the notation of the proof of Lemma 5-1-2,  $\tau_l^{-1}(U_{X_2}) \cap B_{a_i} // \mathbb{C}^* \hookrightarrow U_{X_2} \times (\mathbb{P}^N)_{a_i} // \mathbb{C}^*$  is embedded as a closed analytic subvariety. In fact, let  $\mathcal{I}$  be the ideal sheaf defining  $\tau_l^{-1}(U_{X_2})$  inside of  $U_{X_2} \times \mathbb{P}^N$ . Let  $\mathcal{O}_{\mathbb{P}^N}(1)$  be the very ample sheaf on  $\mathbb{P}^N$ . For sufficiently large  $l \in \mathbb{N}$ , we have an exact sequence

$$\begin{aligned}0 &\rightarrow p_{1*}\{\mathcal{I} \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\} \\ &\rightarrow (p_1)_*\{p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\} = \mathcal{O}_{U_{X_2}} \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l)) \\ &\rightarrow p_{1*}\{\mathcal{O}_{\tau_l^{-1}(U_{X_2})}\} \rightarrow 0\end{aligned}$$

where

$$\begin{aligned}p_{1*}\{\mathcal{I} \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\} \otimes \mathbb{C}(y) &\rightarrow H^0(p_2^{-1}(y), \mathcal{I} \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)|_{p_2^{-1}(y)}) \\ (p_1)_*\{p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\} \otimes \mathbb{C}(y) &\rightarrow H^0(p_2^{-1}(y), p_2^*\mathcal{O}_{\mathbb{P}^N}(l)|_{p_2^{-1}(y)}) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l))\end{aligned}$$

are isomorphisms for any  $y \in U_{X_2}$ . It is easy to see inductively as before that the construction can be carried out so that  $p_{1*}\{\mathcal{I} \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\}$  splits into the eigenspaces with respect to the action of  $\mathbb{C}^*$ . We may also assume that  $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l))^{\mathbb{C}^*}$  generates the homogeneous invariant ring  $\bigoplus_{m \geq 0} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(ml))^{\mathbb{C}^*}$ . Now by Fogarty-Mumford-Kirwan [1] Chapter 1 §2, in each fiber  $p_2^{-1}(y) \cong \mathbb{P}^N$ , the invariant homogeneous polynomials  $H^0(p_2^{-1}(y), \mathcal{I} \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)|_{p_2^{-1}(y)})^{\mathbb{C}^*}$  defines the image of  $B_{a_i}$  under the quotient map  $U_{X_2} \times (\mathbb{P}^N)_{a_i} \rightarrow U_{X_2} \times (\mathbb{P}^N)_{a_i} // \mathbb{C}^*$ . That is to say, the invariant part  $p_{1*}\{\mathcal{I} \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\}^{\mathbb{C}^*}$  defines the image of  $B_{a_i}$  under the quotient map globally inside of  $U_{X_2} \times (\mathbb{P}^N)_{a_i} // \mathbb{C}^* \hookrightarrow U_{X_2} \times \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l))^{\mathbb{C}^*})$ , verifying the assertion.

It is also worthwhile noting that the quotient  $\pi : B_{a_i} \rightarrow B_{a_i} // \mathbb{C}^*$  is characterized by the following three properties:

(i) set-theoretically  $B_{a_i} // \mathbb{C}^*$  is the set of equivalence classes of  $\mathbb{C}^*$ -orbits in  $B_{a_i}$  where two orbits  $O(p)$  and  $O(q)$  are equivalent if and only if  $\overline{O(p)} \cap \overline{O(q)} \neq \emptyset$  (Note that  $B_{a_i}$  is quasi-elementary.),

(ii)  $U \subset B_{a_i} // \mathbb{C}^*$  is open if and only if  $\pi^{-1}(U) \subset B_{a_i}$  is open (with respect to the usual topology), and

$$(iii) \Gamma(U, \mathcal{O}_{B_{a_i} // \mathbb{C}^*}) = \Gamma(\pi^{-1}(U), \mathcal{O}_{B_{a_i}})^{\mathbb{C}^*}.$$

For each  $p \in B_{a_i} - F_{B_{a_i}}^*$  we choose  $\eta_p : U_p \rightarrow V_p$  as constructed in Lemma 5-1-2. Then these  $U_p$  form an open covering of  $B_{a_i}$  such that  $U_p = \pi^{-1}(\pi(U_p))$  by the condition  $(*)$  and that  $\pi(U_p) = U_p // \mathbb{C}^* \xrightarrow{\sim} V_p // \mathbb{C}^*$  is an open subset of an affine (not necessarily nonsingular) toric variety  $X_p // \mathbb{C}^*$ .

Step 3. Torification

Step 4. Recovery from Singular to Nonsingular

The arguments for these two steps work verbatim if we replace Luna's locally toric charts with the  $\mathbb{C}^*$ -equivariant analytic isomorphisms  $\eta_p : U_p \rightarrow V_p \subset X_p$  as described in Lemma 5-1-2.

This completes the proof for the weak factorization theorem for bimeromorphic maps.

§5-2. Equivariant factorization under group action

**Theorem 5-2-1 (Equivariant Weak Factorization Theorem).** *Let  $X_1$  and  $X_2$  be complete nonsingular varieties over an algebraically closed field  $K$  of characteristic zero. Let  $G$  be a group acting on  $X_1$  and  $X_2$ . (Note that the action does NOT have to be over the base field  $K$ .) Let  $\phi : X_1 \dashrightarrow X_2$  be a birational map which is equivariant under the action of  $G$ . Let  $X_1 \supset U \subset X_2$  be a common open set over which  $\phi$  is an isomorphism. Then  $\phi$  can be factored into a sequence of equivariant blowups and blowdowns with smooth  $G$ -invariant centers disjoint from  $U$ . (The center may be reducible, i.e., a collection of several disjoint smooth irreducible components, which is  $G$ -invariant as a whole.) That is to say, there exists a sequence of  $G$ -equivariant birational maps between complete nonsingular varieties with  $G$ -actions*

$$X_1 = V_1 \xrightarrow{\psi_1} V_2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{i-1}} V_i \xrightarrow{\psi_i} V_{i+1} \xrightarrow{\psi_{i+1}} \cdots \xrightarrow{\psi_{l-2}} V_{l-1} \xrightarrow{\psi_{l-1}} V_l = X_2$$

where

- (i)  $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$ ,
- (ii)  $\psi_i$  are isomorphisms over  $U$ , and
- (iii) either  $\psi_i : V_i \dashrightarrow V_{i+1}$  or  $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$  is a morphism obtained by blowing up smooth  $G$ -invariant center disjoint from  $U$ .

Moreover, if both  $X_1$  and  $X_2$  are projective, then we can choose a factorization so that all the intermediate varieties  $V_i$  are projective.

*Proof.*

Again we only specify the modifications we have to make at each step of the strategy for the proof described in Chapter 0. Introduction.

Step 1. Elimination of points of indeterminacy

Though not explicitly stated in Hironaka [2,3] (cf.Bierstone-Milman [1] Encinas-Villamayor [1]), its method for elimination of points of indeterminacy works equivariantly under a group action. Thus Lemma 1-4-2 holds in the equivariant case as well and hence we may assume that  $\phi : X_1 \rightarrow X_2$  is a projective birational morphism which is the blowup of  $X_2$  along a  $G$ -invariant ideal sheaf  $J \subset \mathcal{O}_{X_2}$  (and hence  $\phi$  is  $G$ -equivariant) with the support of  $\mathcal{O}_{X_2}/J$  being disjoint from  $U$ .

### Step 2. Construction of a birational cobordism

Since the canonical resolution of singularities is  $G$ -equivariant by the property ( $\spadesuit^{res} - 1$ ), the construction in Theorem 2-2-2 is also  $G$ -equivariant. We remark that the morphism  $\tau : \overline{B} \rightarrow X_2$  is projective and  $G$ -equivariant as well as  $K^*$ -equivariant. We also remark that a relatively ample line bundle  $\mathcal{L}$  on  $\overline{B}$  over  $X_2$  with a  $K^*$ -linearization (We set  $\mathcal{E} = \tau_*(\mathcal{L}^{\otimes l})$  in the proof of Theorem 2-2-2.) can be chosen so that it comes with a  $G$ -linearization which commutes with the  $K^*$ -linearization. In fact,  $\mathcal{L}$  can be chosen to be of the form  $\mathcal{O}_{\overline{B}}(n \cdot p_2^*(\infty) - E)$  where  $p_2^*(\infty)$  is the pull-back of the hyperplane divisor  $\infty \in \mathbb{P}^1$  and where  $E$  is an exceptional divisor for the birational morphism  $\overline{B} \rightarrow W_0 = X_2 \times \mathbb{P}^1$  and is  $G$ -invariant. We identify the local sections with the elements in the function field

$$\Gamma(U, \mathcal{L}) = \{s_f = f \in K(\overline{B}); \text{div}(f) + n \cdot p_2^*(\infty) - E|_U \geq 0\}$$

and choose the linearizations of  $K^*$  and  $G$  to be the one induced from the actions on the function field  $K(W_0 = X_2 \times \mathbb{P}^1) = K(\overline{B})$ , i.e.,

$$\begin{aligned} t^* s_f &:= t^* f \in \Gamma(t^{-1}(U), \mathcal{L}) \text{ for } t \in K^* \\ g^* s_f &:= g^* f \in \Gamma(g^{-1}(U), \mathcal{L}) \text{ for } g \in G. \end{aligned}$$

As the actions of  $K^*$  and  $G$  on  $X_2 \times \mathbb{P}^1$  obviously commute with each other, so do the linearizations defined as above. It also follows then that the linearization of  $K^*$  on  $\mathcal{L}$  commutes with the twisted linearizations discussed in §2-3. This implies that the quasi-elementary pieces  $B_{a_i}$  are  $G$ -invariant. The rest of the argument goes without any change.

### Step 3. Torification

Since the definition of the torific ideals is canonical in terms of the action of  $K^*$ , which commutes with that of  $G$ , the torific ideals are  $G$ -invariant. Thus the torification is  $G$ -equivariant. The argument in this part goes without any change.

### Step 4. Recovery from Singular to Nonsingular

First we look at Lemma 4-1-2. Note that  $f_{i-} : W_i^{tor} \rightarrow W_i$  is the blowup of the  $G$ -invariant ideal sheaf  $I_{i-} = (f_{i-})_* \mathcal{O}_{W_i^{tor}}(-E_i^{tor})$ , where  $E_i^{tor}$  is a  $G$ -invariant effective Cartier divisor, since the torific ideal is  $G$ -invariant by Step 3. (The same claim holds replacing the negative sign – with the positive sign for the subscripts.) Secondly we look at Proposition 4-1-3. Since the canonical resolution of singularities and the canonical principalization of ( $G$ -invariant) ideals are  $G$ -equivariant, we see that  $h_{i-} : W_{i-}^{can} \rightarrow W_i^{tor}$  and  $h_{i+} : W_{i+}^{can} \rightarrow W_i^{tor}$  are also  $G$ -equivariant. Therefore, we also conclude that  $\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$  is  $G$ -equivariant V-toroidal birational map.

Thus the proof of the equivariant weak factorization theorem is reduced to the following:

**Theorem 5-2-1 (Equivariant Strong Factorization Theorem for Toroidal Birational Maps).** *Let  $(U_{W_1}, W_1)$  and  $(U_{W_2}, W_2)$  be nonsingular toroidal embeddings. Let  $G$  be a group acting on them as automorphisms of toroidal embeddings. Let*

$$\varphi : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$$

*be a proper and toroidal birational map which is  $G$ -equivariant. Then  $\phi$  can be factored into a sequence of toroidal blowups immediately followed by toroidal blowdowns with  $G$ -invariant smooth centers (which may be reducible).*

*Proof.*

First note that the process of Lemma 1-2-6, making a general nonsingular toroidal embedding into the one without self-intersection, is  $G$ -equivariant. Thus we may assume both  $(U_{W_1}, W_1)$  and  $(U_{W_2}, W_2)$  are without self-intersection.

Since  $\varphi$  is toroidal, by Definition 1-2-3 there exists a toroidal embedding  $(U_Z, Z)$  which dominates  $(U_{W_1}, W_1)$  and  $(U_{W_2}, W_2)$  by proper and toroidal birational morphisms

$$(U_{W_1}, W_1) \xleftarrow{f_1} (U_Z, Z) \xrightarrow{f_2} (U_{W_2}, W_2).$$

First by the process of Lemma 1-2-6, we may assume that  $(U_Z, Z)$  is a nonsingular toroidal embedding without self-intersection. By Kempf-Knudsen-Mumford-SaintDonat [1] Abramovich-Karu [1], the morphisms  $f_1$  and  $f_2$  correspond to the refinements of the conical complexes

$$\begin{aligned} \Delta_{f_1} : \Delta_Z &\rightarrow \Delta_{W_1} \\ \Delta_{f_2} : \Delta_Z &\rightarrow \Delta_{W_2}. \end{aligned}$$

Remark that we have a natural homomorphism

$$h : G \rightarrow \text{Aut}(\Delta_{W_1}),$$

sending  $g \in G$ , considered as an automorphism  $g : (U_{W_1}, W_1) \rightarrow (U_{W_1}, W_1)$ , to an automorphism of the conical complex  $h(g) = \Delta_g : \Delta_{W_1} \rightarrow \Delta_{W_1}$ . Since  $\text{Aut}(\Delta_{W_1})$  is a finite group, so is  $\overline{G} := G/\text{Ker}(h)$ . Let  $\Delta_{\tilde{f}_1} : \Delta_{\tilde{Z}} \rightarrow \Delta_{W_1}$  be the smallest common refinement of  $\{\Delta_{\bar{g}} \circ \Delta_{f_1} : \Delta_Z \rightarrow \Delta_{W_1}; \bar{g} \in \overline{G}\}$ . Though  $G$  may not act on the original  $(U_Z, Z)$ , by replacing  $(U_Z, Z)$  with  $(U_{\tilde{Z}}, \tilde{Z})$  associated to the refinement  $\Delta_{\tilde{f}_1} : \Delta_{\tilde{Z}} \rightarrow \Delta_{W_1}$  (and then taking the canonical resolution of singularities, if necessary), we may assume from the beginning that  $G$  acts on the toroidal embedding  $(U_Z, Z)$  and  $f_1 : (U_Z, Z) \rightarrow (U_{W_1}, W_1)$  is  $G$ -equivariant and hence that  $f_2 : (U_Z, Z) \rightarrow (U_{W_2}, W_2)$  is  $G$ -equivariant as well.

Thus we only have to provide factorization for a  $G$ -equivariant toroidal birational morphism  $f : (U_Z, Z) \rightarrow (U_W, W)$  between nonsingular toroidal embeddings without self-intersection, associated to the refinement  $\Delta_f : \Delta_Z \rightarrow \Delta_W$  (cf. the argument in the proof of Theorem 4-2-1). Let  $\overline{G} \subset \text{Aut}(\Delta_W)$  be the subgroup of the automorphism group of the conical complex  $\Delta_W$  induced by the action of  $G$ . Note that  $\overline{G}$  acts on  $\Delta_Z$  as well as on  $\Delta_W$  and that  $\Delta_f$  is  $\overline{G}$ -equivariant.

Now we use an idea of Abramovich-Wang [1]. Take the barycentric star subdivision  $\Delta_b : \Delta_{\hat{W}} \rightarrow \Delta_W$  (cf. Definition 2.1 in Abramovich-Matsuki-Rashid [1]), which is  $\overline{G}$ -equivariant. It is straightforward to see that the corresponding morphism  $b : (U_{\hat{W}}, \hat{W}) \rightarrow (U_W, W)$  between nonsingular toroidal embeddings without

self-intersection is the blowup of a  $G$ -invariant toroidal ideal  $I$  on  $(U_W, W)$  (cf. the argument in the proof of Theorem 2-2-2). Let  $\lambda : \hat{Z} \rightarrow Z$  be the canonical principalization of  $f^{-1}I \cdot \mathcal{O}_Z$ , which is a  $G$ -invariant and toroidal ideal since  $f$  is  $G$ -equivariant and toroidal. Thus  $\lambda : (U_{\hat{Z}} = \lambda^{-1}(U_Z), \hat{Z}) \rightarrow (U_Z, Z)$  is a  $G$ -equivariant and toroidal morphism between nonsingular toroidal embeddings without self-intersection by the property ( $\spadesuit^{can} - 1$ ) of the canonical principalization, and hence so is the induced morphism  $\hat{f} : (U_{\hat{Z}}, \hat{Z}) \rightarrow (U_{\hat{W}}, \hat{W})$ . Let  $\Delta_{\hat{f}} : \Delta_{\hat{Z}} \rightarrow \Delta_{\hat{W}}$  be the associated map of the conical complexes, which is  $\overline{G}$ -equivariant. Observe that, thanks to the process of taking the barycentric star subdivision, the conical complex  $\Delta_{\hat{W}}$  has the following property ( $\natural$ ):

( $\natural$ ) If a cone  $\sigma \in \Delta_{\hat{W}}$  is mapped to itself by some  $\overline{g} \in \overline{G}$ , then  $\overline{g}$  acts as an identity on the cone  $\sigma$ .

Observe also that the conical complex  $\Delta_{\hat{Z}}$  satisfies the same property ( $\natural$ ).

It follows that the quotient  $\Delta_{\hat{W}}/\overline{G}$  (resp.  $\Delta_{\hat{Z}}/\overline{G}$ ) is again a nonsingular conical complex and the projection  $\Delta_{\hat{W}} \rightarrow \Delta_{\hat{W}}/\overline{G}$  (resp.  $\Delta_{\hat{Z}} \rightarrow \Delta_{\hat{Z}}/\overline{G}$ ) maps the cones in  $\Delta_{\hat{W}}$  (resp.  $\Delta_{\hat{Z}}$ ) isomorphically to the cones in  $\Delta_{\hat{W}}/\overline{G}$  (resp.  $\Delta_{\hat{Z}}/\overline{G}$ ) and we have a map of nonsingular conical complexes  $\Delta_{\hat{f}}/\overline{G} : \Delta_{\hat{Z}}/\overline{G} \rightarrow \Delta_{\hat{W}}/\overline{G}$ .

Now apply the combinatorial algorithm of Morelli [1][2] Abramovich-Matsukid-Rashid [1] to have a conical complex  $\Delta_V/\overline{G}$  which is a comon refinement of  $\Delta_{\hat{W}}/\overline{G}$  and  $\Delta_{\hat{Z}}/\overline{G}$

$$\Delta_{\hat{W}}/\overline{G} \xleftarrow{\Delta_{\mu_{\hat{W}}} / \overline{G}} \Delta_V/\overline{G} \xrightarrow{\Delta_{\mu_{\hat{Z}}} / \overline{G}} \Delta_{\hat{Z}}/\overline{G}$$

where  $\mu_{\hat{W}}/\overline{G}$  (resp.  $\mu_{\hat{Z}}/\overline{G}$ ) is a sequence of smooth star subdivisions. We can pull back the refinement, i.e., set

$$\Delta_V = \Delta_{\hat{W}} \times_{\Delta_{\hat{W}}/\overline{G}} \Delta_V/\overline{G} = \Delta_{\hat{Z}} \times_{\Delta_{\hat{Z}}/\overline{G}} \Delta_V/\overline{G}$$

to obtain the  $\overline{G}$ -equivariant maps

$$\Delta_{\hat{W}} \xleftarrow{\Delta_{\mu_{\hat{W}}} } \Delta_V \xrightarrow{\Delta_{\mu_{\hat{Z}}} } \Delta_{\hat{Z}}$$

where  $\mu_{\hat{W}}$  (resp.  $\mu_{\hat{Z}}$ ) is a sequence of smooth star subdivisions, possibly taking several disjoint smooth star subdivisions simultaneously. Geometrically interpreted, we have the assertion of the theorem.

This completes the proof of Theorem 5-2-1 and hence the proof of the equivariant weak factorization theorem.

### Remark 5-2-2.

Equivariant Weak Factorization Theorem for Bimeromorphic Maps can be proved in almost an identical manner, combining the modifications in §5-1 and §5-2. (Note that the relatively (very) ample line bundle  $\mathcal{L} = \mathcal{L}_l$  constructed inductively in §5-1 has not only a  $\mathbb{C}^*$ -linearization but also a natural  $G$ -linearization which commutes with the  $\mathbb{C}^*$ -linearization.) Details are left to the reader as an exercise.

§5-3. Factorization over a non-algebraically closed field

**Theorem 5-3-1 (Weak Factorization Theorem over a NON-algebraically closed field).** Let  $\phi : X_1 \dashrightarrow X_2$  be a birational map between varieties smooth and proper over a field  $K$ , which may not be algebraically closed, of characteristic zero. Let  $X_1 \supset U \subset X_2$  be a common open subset over which  $\phi$  is an isomorphism. Then  $\phi$  can be factored into a sequence of blowups and blowdowns with irreducible centers smooth over  $K$  and disjoint from  $U$ . That is to say, there exists a sequence of bbirational maps between varieties smooth and proper over  $K$

$$X_1 = V_1 \xrightarrow{\psi_1} V_2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{i-1}} V_i \xrightarrow{\psi_i} V_{i+1} \xrightarrow{\psi_{i+1}} \cdots \xrightarrow{\psi_{l-2}} V_{l-1} \xrightarrow{\psi_{l-1}} V_l = X_2$$

where

- (i)  $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$ ,
- (ii)  $\psi_i$  are isomorphisms over  $U$ , and
- (iii) either  $\psi_i : V_i \dashrightarrow V_{i+1}$  or  $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$  is a morphism obtained by blowing up an irreducible center smooth over  $K$  and disjoint from  $U$ .

Moreover, if both  $X_1$  and  $X_2$  are projective, then we can choose a factorization so that all the intermediate complex manifolds  $V_i$  are projective.

*Proof.*

We specify the modifications we have to make at each step of the proof of the strategy for the proof described in Chapter 0. Introduction.

Step 1. Elimination of points of indeterminacy

Hironaka's method for elimination of points of indeterminacy works over ANY field of characteristic zero (cf. Remark 1-4-2). Hence again via Lemma 1-4-2, we may assume that  $\phi : X_1 \rightarrow X_2$  is a projective birational morphism which is the blowup of of  $X_2$  along an ideal sheaf  $J \subset \mathcal{O}_{X_2}$  defined over  $K$  with the support of  $\mathcal{O}_{X_2}/J$  being disjoint from  $U$ .

Step 2. Construction of a birational cobordism

Since the canonical resolution of singularities also works over any field of characteristic zero, the construction of a birational cobordism goes without any change.

Step 3. Torification

Step 4. Recovery from Singular to Nonsingular

Here we carry the argument by taking the base change from the original field  $K$  to its algebraic closure  $\bar{K}$ .

As  $X_1$  and  $X_2$  share a common open subset  $U$ , after base change

$$\begin{aligned} \overline{X_1} &:= X_1 \times_{\text{Spec } K} \text{Spec } \bar{K} \\ (\text{resp. } \overline{X_2}) &:= X_2 \times_{\text{Spec } K} \text{Spec } \bar{K}, \\ \overline{\{B\}} &:= B \times_{\text{Spec } K} \text{Spec } \bar{K} \\ \overline{U} &:= U \times_{\text{Spec } K} \text{Spec } \bar{K} \end{aligned}$$

it splits into a finite number of disjoint components smooth over  $\bar{K}$

$$(\overline{X_1})_{g_j} \quad (\text{resp. } (\overline{X_2})_{g_j}, \quad (\overline{\{B\}})_{g_j}, (\overline{U})_{g_j})$$

such that they are conjugate to each other under the action of the Galois group  $G = \text{Gal}(\overline{K}/K)$  where we set  $G_e$  to be the decomposition group of some component  $(\overline{X}_1)_e$ , i.e.,

$$G_e := \{g \in G; g((\overline{X}_1)_e) = (\overline{X}_1)_e\}$$

and where  $C = \{g_o = e, \dots, g_j, \dots\}$  is a complete representative modulo  $G_e$ , and that  $(\overline{\{B\}})_{g_j}$  is a birational cobordism for the birational map  $(\overline{\varphi})_{g_j} : (\overline{X}_1)_{g_j} \dashrightarrow (\overline{X}_2)_{g_j}$  respecting the open subset  $(\overline{U})_{g_j}$ . (Note that we use the notation  $\overline{\{B\}}$  in order to distinguish it from the compactified birational cobordism  $\overline{B}$ .)

Now we apply the argument for equivariant weak factorization in §5-2 to the birational map  $(\overline{\varphi})_e : (\overline{X}_1)_e \dashrightarrow (\overline{X}_2)_e$  over an algebraically closed field  $\overline{K}$ , induced from the birational cobordism  $(\overline{\{B\}})_e$ , to obtain a  $G_e$ -equivariant factorization

$$\begin{aligned} (\overline{X}_1)_e &= (\overline{V}_1)_e \xrightarrow{(\overline{\psi}_1)_e} (\overline{V}_2)_e \xrightarrow{(\overline{\psi}_2)_e} \\ &\dots \xrightarrow{(\overline{\psi}_{i-1})_e} (\overline{V}_i)_e \xrightarrow{(\overline{\psi}_i)_e} (\overline{V}_{i+1})_e \xrightarrow{(\overline{\psi}_{i+1})_e} \dots \\ &\xrightarrow{(\overline{\psi}_{l-2})_e} (\overline{V}_{l-1})_e \xrightarrow{(\overline{\psi}_{l-1})_e} (\overline{V}_l)_e = (\overline{X}_2)_e \end{aligned}$$

where

- (i)  $(\overline{\phi})_e = (\overline{\psi}_{l-1})_e \circ (\overline{\psi}_{l-2})_e \circ \dots \circ (\overline{\psi}_2)_e \circ (\overline{\psi}_1)_e$ ,
- (ii)  $(\overline{\psi}_i)_e$  are isomorphisms over  $(\overline{U})_e$ , and
- (iii) either  $(\overline{\psi}_i)_e : (\overline{V}_i)_e \dashrightarrow (\overline{V}_{i+1})_e$  or  $(\overline{\psi}_i)_e^{-1} : (\overline{V}_{i+1})_e \dashrightarrow (\overline{V}_i)_e$  is a morphism obtained by blowing up smooth  $G_e$ -invariant center  $(\overline{C}_i)_e$  disjoint from  $(\overline{U})_e$ .

Taking the conjugate induced by the action of the element  $g_j \in G$  in the Galois group, we obtain the corresponding factorization for  $(\overline{\psi}_i)_{g_j} : (\overline{X}_1)_{g_j} \dashrightarrow (\overline{X}_2)_{g_j}$

$$\begin{aligned} (\overline{X}_1)_{g_j} &= (\overline{V}_1)_{g_j} \xrightarrow{(\overline{\psi}_1)_{g_j}} (\overline{V}_2)_{g_j} \xrightarrow{(\overline{\psi}_2)_{g_j}} \\ &\dots \xrightarrow{(\overline{\psi}_{i-1})_{g_j}} (\overline{V}_i)_{g_j} \xrightarrow{(\overline{\psi}_i)_{g_j}} (\overline{V}_{i+1})_{g_j} \xrightarrow{(\overline{\psi}_{i+1})_{g_j}} \dots \\ &\xrightarrow{(\overline{\psi}_{l-2})_{g_j}} (\overline{V}_{l-1})_{g_j} \xrightarrow{(\overline{\psi}_{l-1})_{g_j}} (\overline{V}_l)_{g_j} = (\overline{X}_2)_{g_j}. \end{aligned}$$

Collectively, we obtain the factorization over  $\overline{K}$  for

$$\overline{\varphi} : \overline{X}_1 \dashrightarrow \overline{X}_2$$

into blowups with centers  $\coprod_{g_j \in C} (\overline{C}_i)_{g_j}$ , which are  $G$ -invariant and smooth over  $\overline{K}$ . Therefore, we obtain the factorization over  $K$  for  $\phi : X_1 \dashrightarrow X_2$

$$X_1 = V_1 \xrightarrow{\psi_1} V_2 \xrightarrow{\psi_2} \dots \xrightarrow{\psi_{i-1}} V_i \xrightarrow{\psi_i} V_{i+1} \xrightarrow{\psi_{i+1}} \dots \xrightarrow{\psi_{l-2}} V_{l-1} \xrightarrow{\psi_{l-1}} V_l = X_2$$

where

- (i)  $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \dots \circ \psi_2 \circ \psi_1$ ,
- (ii)  $\psi_i$  are isomorphisms over  $U$ , and

(iii') either  $\psi_i : V_i \dashrightarrow V_{i+1}$  or  $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$  is a morphism obtained by blowing up a center  $C_i$  smooth over  $K$  and disjoint from  $U$ .

In order to guarantee that the centers of blowups to be irreducible and satisfy the condition (iii), instead of blowing up  $C_i$  which may be reducible, we blow up each irreducible component one by one and replace the original sequence with the corresponding refinement.

The “moreover” part concerning the projectivity is also obvious from the construction.

This completes the proof of the weak factorization theorem over a (possibly) NON-algebraically closed field.

#### §5-4. Factorization in the logarithmic category

**Theorem 5-4-1 (Weak Factorization Theorem in the Logarithmic Category).** *Let  $(U_{X_1}, X_1)$  and  $(U_{X_2}, X_2)$  be complete nonsingular toroidal embeddings over an algebraically closed field  $K$  of characteristic zero. Let  $\phi : (U_{X_1}, X_1) \dashrightarrow (U_{X_2}, X_2)$  be a birational map which is an isomorphism over  $U_{X_1} = U_{X_2}$ . Then  $\phi$  can be factored into a sequence of blowups and blowdowns with smooth ADMISSIBLE and irreducible centers disjoint from  $U_{X_1} = U_{X_2}$ . That is to say, there exists a sequence of birational maps between complete nonsingular toroidal embeddings*

$$\begin{aligned} (U_{X_1}, X_1) &= (U_{V_1}, V_1) \xrightarrow{\psi_1} (U_{V_2}, V_2) \xrightarrow{\psi_2} \\ &\dots \xrightarrow{\psi_{i-1}} (U_{V_i}, V_i) \xrightarrow{\psi_i} (U_{V_{i+1}}, V_{i+1}) \xrightarrow{\psi_{i+1}} \dots \\ &\xrightarrow{\psi_{l-2}} (U_{V_{l-1}}, V_{l-1}) \xrightarrow{\psi_{l-1}} (U_{V_l}, V_l) = (U_{X_2}, X_2) \end{aligned}$$

where

- (i)  $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \dots \circ \psi_2 \circ \psi_1$ ,
- (ii)  $\psi_i$  are isomorphisms over  $U_{V_i}$ , and

(iii) either  $\psi_i : (V_i, U_{V_i}) \dashrightarrow (U_{V_{i+1}}, V_{i+1})$  or  $\psi_i^{-1} : (U_{V_{i+1}}, V_{i+1}) \dashrightarrow (U_{V_i}, V_i)$  is a morphism obtained by blowing up a smooth irreducible center disjoint from  $U_{V_i} = U_{V_{i+1}}$  and transversal to the boundary  $D_{V_i} = V_i - U_{V_i}$  or  $D_{V_{i+1}} = V_{i+1} - U_{V_{i+1}}$ , i.e., at each point  $p \in V_i$  or  $p \in V_{i+1}$  there exists a regular coordinate system  $\{x_1, \dots, x_n\}$  in a neighborhood  $p \in U_p$  such that  $D_{V_i} \cap U_p$  (or  $D_{V_{i+1}} \cap U_p$ ) =  $\{\prod_{j \in J} x_j = 0\}$  and  $C_i \cap U_p = \{\prod_{j \in J \cup J'} x_j = 0\}$  for some subsets  $J, J' \subset \{1, \dots, n\}$ .

Moreover, if both  $(U_{X_1}, X_1)$  and  $(U_{X_2}, X_2)$  are projective, then we can choose a factorization so that all the intermediate toroidal embeddings  $(U_{V_i}, V_i)$  are projective.

*Proof.*

We specify the modifications we have to make at each step of the proof of the strategy for the proof described in Chapter 0. Introduction.

#### Step 1. Elimination of points of indeterminacy

Lemma 1-4-1 holds in the above-mentioned logarithmic category where all the centers of the blowups necessary for elimination of points of indeterminacy can be taken to be admissible as well as smooth. Thus we may assume that  $\phi : (U_{X_1}, X_1) \rightarrow (U_{X_2}, X_2)$  is a projective morphism which is the blowup of  $X_2$  along an ideal sheaf  $J \subset \mathcal{O}_{X_2}$  with the support of  $\mathcal{O}_{X_2}/J$  being disjoint from  $U_{X_2} = U_{X_1}$ .

### Step 2. Construction of a birational cobordism

In an identical manner to the proof of Theorem 2-2-2, we can construct a compactified birational cobordism  $\tau : (U_{\overline{B}}, \overline{B}) \rightarrow (U_{X_2}, X_2)$  for  $\phi$  projective over  $X_2$  such that  $\tau^{-1}(D_{X_2}) = D_{\overline{B}}$  is a divisor with only normal crossings where  $D_{X_2} = X_2 - U_{X_2}$  and  $D_{\overline{B}} = \overline{B} - U_{\overline{B}}$  and that

$$(U_B, B)_+ / K^* = (U_{X_2}, X_2)$$

$$(U_B, B)_- / K^* = (U_{X_1}, X_1)$$

as toroidal embeddings.

When we construct a locally toric chart for a point  $p \in B$  (Though  $(U_B, B)$  is a nonsingular toroidal embedding, the action of  $K^*$  may not be toroidal. Thus we still call the chart a locally toric chart), we use the method (A) in Proposition 1-3-4 so that  $U_p \cap D_B = V_p \cap D_B$  is included in  $\eta_p^{-1}(D_{X_p})$ , where  $D_B = B - U_b$  and  $D_{X_p} = X_p - U_{X_p}$ . This can be done, in the notation of the proof (A) of Proposition 1-3-4, by choosing a regular system of parameters

$$f_1, \dots, f_n \in m_q$$

around the limit point  $q \in \overline{O(p)}$ , consisting of eigenfunctions, so that each irreducible component  $D$  of  $V_q \cap D_{\overline{B}}$  is defined by some coordinate among  $f_1, \dots, f_n$  (shrinking  $V_q$  if necessary). In fact, since the whole boundary  $D_{\overline{B}}$  is invariant under the action of  $K^*$ , so is  $D$  and hence its defining ideal  $I_D \subset A(V_q)$  is also  $K^*$ -invariant. Since  $K^*$  is reductive, the ideal  $I_D$  splits into the eigenspaces according to the characters under the action of  $K^*$ . As  $D$  is nonsingular at  $q$ , there exists an eigenfunction  $f_D \in I_D$  and a coordinate, say  $f_1$ , such that

$$f_D \bmod m_q = f_1 \bmod m_q.$$

By replacing  $f_1$  with  $f_D$  in the regular system of parameters, we achieve the goal.

### Step 3. Torification

#### Step 4. Recovery from Singular to Nonsingular

In the torification of the quasi-elementary cobordism  $B = B_{a_i}$ , after blowing up the torific ideal to obtain  $\mu : B^{tor} \rightarrow B$ , we only have to show that

$$(B^{tor} - \{D^{tor} \cup \mu^{-1}(D_B)\}, B^{tor})$$

has a toroidal structure with respect to which the induced action of  $K^*$  is toroidal. But this follows immediately, since already

$$(B^{tor} - D^{tor}, B^{tor})$$

has a toroidal structure with respect to which the induced action of  $K^*$  is toroidal and since the additional boundary divisors coming from  $D_B$  coorespond to the toric coordinate divisors in the locally toric charts chosen as above. (In the language of Torification Principle for Toric Varieties, we do NOT have to “remove” the divisors coming from  $D_B$  from the boundary.) The argument for recovery from Singular to Nonsingular also goes without any change, considering the toroidal structures with the divisors coming from  $D_B$  added to the boundaries.

This completes the proof of the weak factorization theorem in the logarithmic category.

**Corollary 5-4-2.** *The Hodge numbers are birational invariants for nonsingular minimal models, i.e., if  $X_1$  and  $X_2$  are projective nonsingular varieties over  $\mathbb{C}$  whose canonical divisors are nef and if they are birational to each other, then we have*

$$h^{p,q}(X_1) = h^{p,q}(X_2) \quad \forall p, q.$$

*Proof.*

This application of the weak factorization theorem to a theorem of Batyrev was communicated to us by J. Denef, F. Loeser and W. Veys. It is easy to see that  $X_1$  and  $X_2$ , being birational and minimal, are isomorphic outside of some codimension 2 loci. By some sequences of blowups with admissible (in the usual sense) centers over the specified loci, we can reach the situation of Theorem 5-4-1, where again the birational map is factored as a sequence of blowups and blowdowns with smooth and admissible (in the sense of Theorem 5-4-1) centers over the same specified loci. Then it is straightforward to see the required invariance of the Hodge numbers via the invariance of Batyrev's stringy  $E$ -function through such blowups and blowdowns with admissible centers. We refer the details of the argument to the papers in the related field (cf. Batyrev [1] Denef-Loeser [1] Kollar [2] Wang [1]).

There is also a slightly different approach to the above theorem by considering the Grothendieck group of the Hodge structures and looking for some elementary but explicit invariant form within the group under blowups and blowdowns with admissible centers (cf. Arapura-Matsuki [1]).

**Remark 5-4-3.**

(i) We can also combine some of the generalizations above, as in Remark 5-2-2, and prove, e.g.,

- an equivariant version of the weak factorization theorem in the logarithmic category or in the usual category over a (possibly) NON-algebraically closed field of characteristic zero

- the weak factorization theorem of bimeromorphic maps in the logarithmic category with or without a group action in consideration.

The modifications we have to make are simple combinations of those discussed in §5-1 through §5-4 and we leave the details to the reader.

(ii) One can try to present the generalizations in §5-1 through §5-4 in a more unified manner, including the case of algebraic spaces (cf. Section 5 in Abramovich-Karu-Matsuki-Włodarczyk [1]). We chose, however, our rather repetitive manner in order to emphasize minor but subtle differences.

## CHAPTER 6. PROBLEMS

In this chapter, we discuss a couple of problems related to our proof of the weak factorization theorem for birational maps.

### §6-1. Effectiveness of the construction

As our proof factorizes a given birational map explicitly via the birational cobordism, it is not merely existential but also constructive. On the other hand, if one asks the following effectiveness question, our construction falls short of giving an affirmative answer at several places.

**Question 6-1-1.** *Suppose that nonsingular projective varieties  $X_1 \subset \mathbb{P}^{n_1}, X_2 \subset \mathbb{P}^{n_2}$  as well as a birational map between them  $\phi : X_1 \dashrightarrow X_2$  are given by a specific set of equations in terms of homogeneous coordinates of the projective spaces. Can we give a factorization of  $\phi$  into blowups and blowdowns with smooth centers in an effective way from the given set of equations?*

We list the places where our construction falls short of being effective (or rather to say, the places where the AUTHOR does not know how to carry out the process in an effective way).

Step 1: Reduction to the case where  $\phi$  is a projective birational morphism

Hironaka's method of elimination of the points of indeterminacy in Lemma 1-4-1

- the construction of the ideal  $I$  in Remark 1-4-2 on  $X_1$  whose blowup factors through  $X_2$  (after taking the graph of  $\phi$  and assuming  $\phi^{-1}$  is a morphism)
- canonical principalization of the ideal  $I$  (Canonical resolution of singularities as well as canonical principalization of ideals are constructive as presented in Bierstone-Milman [1] Villamayor [1] Encinas-Villamayor [1], while the author does not know if their algorithms can be made effective.)

Step 2: Factorization into locally toric birational maps

Construction of a birational cobordism in Theorem 2-2-2

- the construction of the ideal  $J \subset \mathcal{O}_{X_2}$  whose blowup gives the projective morphism  $\phi : X_1 \rightarrow X_2$ , more specifically finding the generators of  $J$  in terms of the homogeneous coordinates for  $X_2$
- canonical resolution of singularities
- the construction of a relatively ample line bundle  $\mathcal{L}$  on  $\overline{B}$ , equipped with a  $K^*$ -action so that for sufficiently large  $l \in \mathbb{N}$  the direct image sheaf  $\mathcal{E} = \tau_*(\mathcal{L}^{\otimes l})$  has the decomposition into the eigen spaces  $\mathcal{E} = \bigoplus_{b \in \mathbb{Z}} \mathcal{E}_b$  and that the compactified birational cobordism  $\overline{B}$  has an equivariant embedding  $\overline{B} \hookrightarrow \mathbb{P}(\mathcal{E})$

Actually the effectiveness for the above could be an immediate consequence of the effectiveness for canonical resolution. In fact, in the process of canonical resolution using any one of the algorithms given by Bierstone-Milman [1] Villamayor [1] Encinas-Villamayor [1]

$$\overline{B} = W_m \rightarrow \cdots \rightarrow W_{i+1} \xrightarrow{\mu_i} W_i \rightarrow \cdots \rightarrow W_1 = X_2 \times \mathbb{P}^1$$

if we can find inductively an equivariant embedding

$$W_i \subset X_2 \times \mathbb{P}^{k_i} \subset \mathbb{P}^{n_2} \times \mathbb{P}^{k_i}$$

where  $K^*$  acts only on the second factor  $\mathbb{P}^{k_i}$  with the semi-invariant homogeneous coordinates, which induces a linearization on  $\mathcal{L}_i = p_2^*\mathcal{O}_{\mathbb{P}^{k_i}}(1)$ , and if we can find effectively some generators consisting of bihomogeneous polynomials (in terms of homogeneous coordinates for  $\mathbb{P}^{n_2}$  and  $\mathbb{P}^{k_i}$ ) of some fixed bidegree for the ideal of the center of blowup for  $\mu_i$ , which is  $K^*$ -invariant, then it is easy to realize the inductive situation for  $W_{i+1}$ . Then we can set  $\mathcal{L} = \mathcal{L}_m$  and  $l = 1$ .

- how to find Luna’s locally toric charts as described in Lemma 2-4-4 satisfying the condition  $(\star)$

### Step 3: Factorization into toroidal birational maps

#### Torification

Once Luna’s toric charts satisfying the condition  $(\star)$  are chosen, all the analysis is reduced to the toric case, where the process of torification is effective as well as constructive.

### Step 4: Recovery of nonsingularity

Again via Luna’s locally toric (toroidal) charts all the analysis here is reduced to the toric case, where the process, including the canonical resolution of singularities and canonical principalization of (toric) ideals, is effective as well as constructive.

### Step 5. Factorization of toroidal birational maps among nonsingular toroidal embeddings

#### Morelli’s combinatorial algorithm

There are two places where the algorithm refers to some existential theorems in Morelli [1][2] Abramovich-Matsuki-Rashid [1].

- Toric Moishezon Theorem

Though we refer to Hironaka’s elimination of points of indeterminacy in the two papers above, the algorithm in Deconcini-Procesi [1] for toric birational maps is effective as well as constructive.

- Sumihiko’s equivariant completion theorem

In the toric case, “filling up” cones between the lower face and upper face of the “vacuum” for the fan representing the birational cobordism can be made effective as well as effective as in Matsuki-Rashid [1].

### §6-2. Toroidalization Conjecture and Strong Factorization Conjecture

**Conjecture 6-2-1 (Toroidalization Algorithm Conjecture).** *Let  $f : (U_X, X) \rightarrow (U_Y, Y)$  be a proper (not necessarily birational) morphism between nonsingular toroidal embeddings such that  $f^{-1}(Y - U_Y) = X - U_X$  and that  $f|_{U_X} : U_X \rightarrow U_Y$  is smooth. Then there should exist an algorithm which specifies the way to take sequences of blowups  $\sigma_X : (U_{X'}, X') \rightarrow (U_X, X)$  and  $\sigma_Y : (U_{Y'}, Y') \rightarrow (U_Y, Y)$  with smooth (and possibly admissible (!?)) centers in the boundaries so that the resulting morphism between nonsingular toroidal embeddings  $\phi' : (U_{X'}, X') \rightarrow (U_{Y'}, Y')$  is toroidal.*

It is easy to see that via the method of elimination of points of indeterminacy and (canonical) resolution of singularities the strong factorization conjecture follows immediately from the toroidalization conjecture.

As we briefly said in the introduction, a morphism between toroidal embeddings

being TOROIDAL (cf. Abramovich-Karu [1] Karu [1]) is equivalent to its being log-smooth (cf. Kato [1] Iitaka [1]). Thus the above conjecture asserts that in the logarithmic category of toroidal embeddings we should have an algorithm which resolves the singularities of a morphism  $\phi : X \rightarrow Y$ , much like the usual ones which resolves the singularities of varieties. The algorithms for resolution of singularities by Bierstone-Milman [1] Villamoyor [1] Encinas-Villamayor [1] and others take advantages of their internal inductional structures together with some ingenious introduction of invariants measuring how singular your variety is. Our hope here is that, going away from the original factorization problem which only applies to the birational maps, via the flexibility of the toroidalization conjecture dealing with morphisms between toroidal embeddings of any dimensions we should start seeing the internal inductional structure with some invariants measuring how singular your morphism is.

Though in dimension 2 we can check the toroidalization conjecture as in Abkult King [1] Abramovich-Karu [1] Abramovich-Karu-Matsuki-Włodarczyk [1] Karu [1] and Cutkosky-Piltant [1], none of the existing methods fully reveals this (conjectural) inductional structure and hence yields a proof even in dimension 3. Probably the approach of Cutkosky-Piltant [1] is the closest to our hope in spirit and the recent solution by Cutkosky [4] to the case where  $\dim X = 3$  and  $\dim Y = 2$  should provide us more information toward the general solution.

Instead of starting from a morphism between nonsingular toroidal emebeddings, we could restate the toroidalization conjecture for a proper morphism  $\phi : X \rightarrow Y$  between any nonsingular varieties claiming after appropriate sequences of blowups  $\sigma_X : X' \rightarrow X$  and  $\sigma_Y : Y' \rightarrow Y$  the resulting morphism  $\phi' : (U_{X'}, X') \rightarrow (U_{Y'}, Y')$  is toroidal for some appropriate choice of open subsets  $U_{X'}$  and  $U_{Y'}$ . In this restated form, the toroidalization conjecture in the case when  $\dim Y = 1$  is (almost) equivalent to resolution of singularities of hypersurface singularities, revealing another close connection between the problem of toroidalization and that of resolution of singularities. (Note that any proper morphism between NONSINGULAR toroidal embeddings stated in Conjecture 6-2-1 is automatically toroidal in the case when  $\dim Y = 1$  and hence trivial.)

Seeing a solution to the (weak) factorization problem of birational maps is hardly an end to this exciting field surrounding resolution of singularities, toroidalization, factorization, semistable reduction, log category ... etc. and we feel that we just had a sneak preview of what is still to come.

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